STOP

It is strongly recommended that you not review these solutions until you have (a) studied first and (b) gotten as far as you could with these problems by yourself. Just looking at the answers is not the same as doing them on your own.
1. Evaluate a function of more than one variable at a specified input or solve for a specified output.

Example: Evaluate $G(2, 3, -4)$, where
$$G(x, y, z) = \frac{x^2 + y^2 - z^2}{x + y + z}.$$ 
We get
$$G(2, 3, -4) = \frac{(2)^2 + (3)^2 - (-4)^2}{2 + 3 + (-4)} = -3.$$

2. Interpret the input and output of a function of more than one variable in context.

Example: The density of votes in support of a politician has been approximated by the function $V(x, y) = 4x^2 + 0.4y + 0.6xy^2$ hundred votes per square mile $x$ miles east and $y$ miles north of her campaign office. At a fundraiser somewhere north and 3 miles east of the campaign office, the density of votes for the politician is 66.4 hundred votes per square mile. How far north of the campaign office is the fundraiser being held?

$$V(3, y) = 4(3)^2 + 0.4(y) + 0.6(3)(y)^2 = 66.4 \text{ hundred votes per square mile.}$$
Solving this with factoring or the quadratic formula gives us
$$y = 4 \text{ or } y = -38/9.$$
We are told that the fundraiser is somewhere to the north, so we disregard the negative option and claim that since $y = 4$, the fundraiser is being held 4 miles north of the campaign office.

3. Model a function of more than one variable given a written description.

Example: A factory maintains inventory which consists of two products, A and B. The factory pays an annual cost to purchase, ship, and store the materials. Each shipment consists of $x$ units of product A and $y$ of product B. Every year, it costs 3 thousand dollars to store each unit of product A and 4 thousand dollars to store each unit of product B. Purchasing costs for an entire order are 250 thousand dollars each year. Shipping costs are $\frac{15000}{x + y}$ thousand dollars each year. Write a formula for $C(x, y)$, the total cost (in thousands) to store, ship, and purchase both products each year.

$$C(x, y) = C_{\text{store}} + C_{\text{ship}} + C_{\text{purchase}} = (3x + 4y) + \frac{15000}{x + y} + 250$$
4. Identify the largest possible domain of a function of two variables in terms of a region in the plane.

**Example:** Find the largest possible domain of \( f(x, y) = \frac{\ln(y-x)}{x+y-2} \).

The argument of the natural logarithm needs to be positive, so \( y-x > 0 \), or \( y > x \). We also need the denominator to be non-zero, so \( x+y-2 \neq 0 \), or \( y \neq 2-x \).

So we need all points \((x, y)\) such that \( y > x \) and \( y \neq 2-x \).

Geometrically, we could describe this domain as the set of all points in the \( xy \)-plane which are both above the line \( y = x \) and not on the line \( y = 2-x \).

5. Sketch graphs of slices (traces) and level curves of functions of two variables.

**Example:** Describe the graph of the \( y = 2 \) trace for \( f(x, y) = 2x - 4 + (y-3)^2 + y \).

The \( y = 2 \) trace corresponds to 
\[
z = 2x - 4 + (2-3)^2 + 2 = 2x - 1.
\]

This is the graph of a line with slope 2 and \( z \)-intercept of \(-1\).

6. Compute first and second partial derivatives of a function of more than one variable.

**Example:** Compute \( f_{ab}(1, -1) \) for function \( f(a, b) = a - be^{-3ab} \).

Begin with
\[
f_a = 1 - be^{-3ab} \cdot (-3b).
\]
\[
= 1 + 3b^2e^{-3ab}
\]

Then we take the derivative of this expression with respect to \( b \) (which requires product rule):
\[
f_{ab} = 0 + \frac{\partial}{\partial b}(3b^2) \cdot e^{-3ab} + 3b^2 \cdot \frac{\partial}{\partial b}(e^{-3ab})
\]
\[
= 6be^{-3ab} + 3b^2 \cdot -3ae^{-3ab}
\]
\[
= 6be^{-3ab} - 9ab^2e^{-3ab}
\]

So finally
\[
f_{ab}(1, -1) = 6(-1)e^{-3(1)(-1)} - 9(1)(-1)^2e^{-3(1)(-1)} = -6e^3 - 9e^3 = -15e^3.
\]

7. Interpret partial derivatives in the context of an applied function.

**Example:** The density of votes in support of a politician has been approximated by the function 
\( V(x, y) = 4x^2 + 0.4y + 0.6xy^2 \) hundred votes per square mile \( x \) miles east and \( y \) miles north of her campaign office. interpret the value of \( V_x(-1, -1) \) in context. Include units.

\( V_x = 8x + 0.6y^2 \). Then \( V_x(-1, -1) = 8(-1) + 0.6(-1)^2 = -7.4 \). This means that one mile west and one mile south of the city center, the density of votes is decreasing at a rate of 7.4 hundred votes per square mile per mile moved eastward.
8. Use partial derivatives to determine whether two goods are substitutes, complementary goods, or neither substitute nor complementary.

We need to compute \( N_y \) and \( M_x \), the rates of change in each product with respect to a change in the other product’s price. If both are negative, then the products are complementary. If both are positive, then the products are substitutes. Otherwise, they are neither complementary nor substitutes.

\[
N_y = 0 - 0 - 12(y + 1)^{-2} = -12(y + 1)^{-2},
\]

which is always negative. Then we find

\[
M_x = 0 - e^{-0.1x} \cdot (-0.1) - 0 = 0.1e^{-0.1x},
\]

which is always positive. Thus, the products must be neither substitutes nor complementary.

9. Calculate the rate of change in a function using the chain rule for a function of two variables.

By the chain rule,

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.
\]

Let \( f(x, y) = 0.1e^x + 8xy \), with \( x = 5 - t \) and \( y = t^2 \). Compute \( \frac{df}{dt} (2) \).

Then we can determine when \( t = 2 \), we get \( x = 5 - 2 = 3 \), and \( y = (2)^2 = 4 \). Thus

\[
\frac{df}{dt} (2) = (0.1e^3 + 8(4)) \cdot (-1) + (8 \cdot 3) \cdot (2 \cdot 2)
\]

\[
\approx 61.99
\]

10. Find the critical points of a function of two variables.

We find critical points of a function of two variables by setting both first partials equal to 0 and solving the resulting system of equations

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 6xy - 3x^2 = 0 \\
\frac{\partial f}{\partial y} &= 3x^2 + 2y = 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 3x(2y - x) = 0 \\ 3x^2 = -2y \end{cases} \Rightarrow \begin{cases} x = 0 \text{ or } x = 2y \\ -3x^2 = y \end{cases}
\]

Using \( x = 0 \) plugged into the second equation, we get \( y = -\frac{3}{2}(0)^2 = 0 \), so one critical point is \((0,0)\).

Using \( x = 2y \) plugged into the second equation, we get
\[-\frac{3}{2}(2y)^2 = y\]
\[-6y^2 = y\]
\[0 = y + 6y^2\]
\[0 = y(1 + 6\quad y)\]

We get that
\[y = 0\text{ or } y = -1/6.\]

That means we have critical points at \(x = 2y = 2(0) = 0\), or \((0, 0)\) (which we already knew), and \(x = 2y = 2(-1/6) = -1/3\), or \((-1/3, -1/6)\), which is new.

**Example:** The function \(z = f(x, y)\) graphed below has critical points at \((0, 0)\) and \((1, 0)\). Classify these points as relative maxima, relative minima, or saddle points. (Note that in this plot, the larger the \(z\) value, the lighter the image)

11. For a function of two variables, identify extrema from its graph.

At each critical point, the graph curves upward along one axis and downward along another, so the points are neither relative maxima nor minima, and thus must be saddle points.

12. Classify (as saddle, relative maximum, or relative minimum) the critical points of a function of two variables.

**Example:** Where, if anywhere, is the relative minimum of \(f(x, y) = -x^2 + 4x - y^2 - 12y\) located?
Nowhere (no relative minimum). \(f\) has a critical point at the solution to \(f_x = 0\) and \(f_y = 0\), which is \((2, -6)\). However, by the second partials test, \(f_{xx}f_{yy} - [f_{xy}]^2 = -2 \cdot -2 - [0]^2 > 0\), with \(f_{xx} < 0\), so the critical point is actually the location of a relative *maximum*.

**Additional Free Response-Style Questions**
13. A utility function, $U(x, y)$, models percent satisfaction with life is given as a function of annual income, $x$, in thousands of dollars and of age, $y$, in years from birth. Write a sentence interpreting the equation $U_x(30, 40) = -0.1$.

The expression $U_x$ represents the rate of change in $U$ as $x$ changes, so: “when income is 30 thousand dollars and the individual is 40 years old, utility is decreasing at a rate of 0.1 percent satisfaction per additional thousand dollars of income.”

14. Find the critical points for $f(x, y) = x^2 y - 3y + 4x$. Classify each as a relative maximum, relative minimum, or saddle point.

We find critical points of a function of two variables by setting both first partials equal to 0 and solving the resulting system of equations

\[
\begin{align*}
  f_x &= 2xy + 4 = 0 \\
  f_y &= x^2 - 3 = 0 \\
\end{align*}
\]

\[\implies \begin{align*}
  2xy &= -4 \\
  x^2 &= 3
\end{align*} \implies \begin{align*}
  y &= \frac{-4}{2x} \\
  x &= \pm \sqrt{3}
\end{align*}
\]

Plugging $x = \pm \sqrt{3}$ into the first equation, we get $y = -2/(\pm \sqrt{3})$, so we get two critical points $(\sqrt{3}, -2/\sqrt{3})$ and $(-\sqrt{3}, 2/\sqrt{3})$.

To classify them, we need to find the second partials:

\[f_{xx} = 2y, \quad f_{yy} = 0, \quad \text{and} \quad f_{xy} = 2x.\]

Now compute $D$ at each point

\[
D(\sqrt{3}, -2/\sqrt{3}) = f_{xx}(\sqrt{3}, -2/\sqrt{3}) \cdot f_{yy}(\sqrt{3}, -2/\sqrt{3}) - [f_{xy}(\sqrt{3}, -2/\sqrt{3})]^2
\]

\[= 2(-2\sqrt{3}) \cdot 0 - [2(\sqrt{3})]^2
\]

\[= -12 < 0
\]

Thus the point $(\sqrt{3}, -2/\sqrt{3})$ is a saddle point.

\[
D(-\sqrt{3}, 2/\sqrt{3}) = f_{xx}(-\sqrt{3}, 2/\sqrt{3}) \cdot f_{yy}(-\sqrt{3}, 2/\sqrt{3}) - [f_{xy}(\sqrt{3}, 2/\sqrt{3})]^2
\]

\[= 2(2\sqrt{3}) \cdot 0 - [2(-\sqrt{3})]^2
\]

\[= -12 < 0
\]

Thus the point $(-\sqrt{3}, 2/\sqrt{3})$ is also a saddle point.
15. A paint company makes two brands of latex paint. Sales figures indicate that if the first brand is sold for \( x \) dollars per quart and the second for \( y \) dollars per quart, the monthly demand for the second brand will be \( Q \) quarts, where

\[
Q(x, y) = 200 + 10x^2 - 20y.
\]

It is estimated that \( t \) months from now the price of the first brand will be \( X(t) = 18 + 0.02t \) dollars per quart and the price of the second will be \( Y(t) = 21 + 0.4\sqrt{7} \) dollars per quart. At what rate will the demand for the second brand of paint be changing with respect to time 9 months from now?

The rate of change with respect to time uses the chain rule for partial derivatives:

\[
\frac{dQ}{dt} = \frac{\partial Q}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Q}{\partial y} \cdot \frac{dy}{dt} = Q_x \cdot X'(t) + Q_y \cdot Y'(t)
\]

\[
\frac{dQ}{dt} \bigg|_{t=9} = [20x] \cdot 0.02 + [-20] \cdot 0.2t^{-1/2}
\]

\[
\approx 5.94
\]

Demand for the second brand will be increasing at a rate of approximately 5.94 quarts per month.

16. A company obtains the licensing rights to a brand for which \( x \) uses by other businesses domestically cost \( 150 - \frac{x}{6} \) thousand dollars and \( y \) uses by other businesses abroad cost \( 100 - \frac{y}{20} \) thousand dollars. The company pays a fixed amount for the costs associated with obtaining the licensing rights. How many uses should the company obtain domestically and abroad to maximize its profit? (Use calculus to verify that you have found a maximum)

The total profit function looks like \( P = \text{Revenue} - \text{Cost} \), but cost is a fixed (unknown) value, so let’s leave it as constant \( C \). Revenue is the sum of the revenues generated from domestic uses and uses abroad, and each of those is the product of price and quantity. So we get

\[
R = x \cdot \left(150 - \frac{x}{6}\right) + y \cdot \left(100 - \frac{y}{20}\right) = 150x - \frac{1}{6}x^2 + 100y - \frac{1}{20}y^2.
\]

We maximize \( P = 150x - \frac{1}{6}x^2 + 100y - \frac{1}{20}y^2 - C \) by first finding any critical points:

\[
\begin{cases}
   P_x(x, y) = 150 - \frac{1}{3}x = 0 \\
   P_y(x, y) = 100 - \frac{1}{10}y = 0
\end{cases}
\implies \begin{cases}
   x = 450 \\
   y = 1000
\end{cases}
\]

We can verify that this corresponds to a maximum by computing the value of the Hessian at \((450, 1000)\), but first we compute the second partial derivatives:

\[
P_{xx}(x, y) = -\frac{1}{3}, \quad P_{yy}(x, y) = -\frac{1}{10}, \quad \text{and} \quad P_{xy}(x, y) = 0
\]

So then

\[
D(450, 1000) = \frac{1}{3} \cdot -\frac{1}{10} = \frac{1}{30}
\]

Our \( D \) is positive, and \( P_{xx} < 0 \), so the point \((450, 1000)\) corresponds to a relative maximum of the profit function. Thus the company should license 450 uses domestically and 1000 uses abroad.
17. Let the function $C(s, h) = s^2 + 6h$ give the approximate cost, in thousands of dollars, to construct a commercial building with a square base $s$ feet on a side, that is $h$ feet high. Sketch the level curve $C(s, h) = 1200$ in the $sh$-plane and interpret it in context.

The equation for the level curve looks like

$$1200 = s^2 + 6h,$$

which we can solve for $h$:

$$1200 = s^2 + 6h$$
$$-6h = s^2 - 1200$$
$$h = -\frac{1}{6}s^2 + 200$$

Which is a downward-opening parabola having vertical axis intercept of 200:

To interpret this level curve, we can say that $C(s, h) = 1200$ implies that we are graphing the combination of side lengths and heights that correspond to a fixed cost of $1200$ thousand (i.e. $1.2$ million) to construct the building.
18. The picture below is a contour plot of a function \( h(x, y) \) of two variables \( x \) and \( y \). The lighter shaded regions represent larger values of the function. The function has two critical points in the region shown. For each of these critical points, give its approximate coordinates and classify it as a relative maximum, relative minimum, or a saddle point of \( h \).

![Contour Plot](image)

There is clearly a relative maximum somewhere inside the oval contour line at the upper left. The center looks like it is at about \((-1.3, 1.3)\), so we say that the function shown has a critical point at about \((-1.3, 1.3)\) and that this critical point is a relative maximum. [For the formula actually used to make the graph, the critical point is at exactly \((-\frac{4}{3}, \frac{4}{3})\).]

There is also a saddle point where the contour lines appear to cross, near \((0, 0)\). To see this, if you follow the line \( y = x \) from the bottom of the area shown diagonally up and right to the right hand side, the function \( h(x, y) \) increases until you reach (about) \((0, 0)\) and then decreases. If you follow the line \( y = -x \) from the top left corner to the bottom right corner, you go up, over the relative maximum discussed above, then down until you get to (about) \((0, 0)\), at which point you start going up again. This means that if you follow the line \( y = x \), you go over a pass at (about) \((0, 0)\). So function shown has a critical point at about \((0, 0)\) and this critical point is a saddle point. [For the formula actually used to make the graph, the critical point is at exactly \((0, 0)\).]

19. Sketch the level curve at \( z = 7 \) of \( z = H(x, y) = (x - 1)^2 + (y - 2)^2 + 6 \).

We need to plot the curve \((x - 1)^2 + (y - 2)^2 = 7\), that is, \((x - 1)^2 + (y - 2)^2 = 1\). This is a circle with radius 1 and center \((1, 2)\). It is the circle in the graph below.

![Level Curve](image)