Guide for Section 7.3: Optimization for Two Variables, part 2

**Ex 4** (Problem 45 in Section 7.3 of the book.) Alternative forms of a gene are called *alleles.* Three alleles, designated A, B, and O, determine the four human blood types A, B, O, and AB. Suppose that \( p \), \( q \), and \( r \) are the proportions of A, B, and O in a particular population, so that \( p + q + r = 1 \). Then, according to the Hardy-Weinberg law in genetics, the proportion of individuals in the population who carry two different alleles is \( P = 2pq + 2pr + 2qr \). What is the largest possible value of \( P \)?

(Alternatively, assume that the relative maximum is a global maximum.)

(Note: The combinations AO and BO give blood types A and B.)

\[
\begin{align*}
    r &= 1 - p - q \\
    P(p, q) &= 2pq + 2p(1-p-q) + 2q(1-p-q) \\
    &= 2p + 2q - 2p^2 - 2pq + 2q - 2rp - 2rq \\
    &= 2p + 2q - 2p^2 - 2pq - 2q^2. \\
    \frac{\partial P}{\partial p} &= 2 - 4p - 2q = 0 \\
    \frac{\partial P}{\partial q} &= 2 - 2p - 4q = 0 \\
\end{align*}
\]

Solve: \( 2 - 4p - 2q = 0 \)
\[ 2 - 2p - 4q = 0 \]

Put \( \dot{q} = 1 - 2p - q = 0 \)
\[
\begin{align*}
    q &= 1 - 2p \\
    2p - 2q &= 0 \\
\end{align*}
\]

Sub: \( 2pq - 2(1-2p) = 0 \)
\[
\begin{align*}
    -1 + 3p &= 0 \\
    p &= \frac{1}{3} \quad \text{Put in last eqn:} \\
    1 - 2(\frac{1}{3}) - q &= 0, \quad \text{so} \quad q = \frac{1}{3}. \\
\end{align*}
\]

Need to check what kind of critical point.

\[
\begin{align*}
    P_{pp}(\frac{1}{3}, \frac{1}{3}) &= -4 \\
    P_{pq}(\frac{1}{3}, \frac{1}{3}) &= -4 \\
    P_{qq}(\frac{1}{3}, \frac{1}{3}) &= -2. \\
\end{align*}
\]

So, rel min \( \frac{1}{3}, \frac{1}{3} \), since \( P_{pp}(\frac{1}{3}, \frac{1}{3}) < 0 \), has a rel. max. 

\[
D(\frac{1}{3}, \frac{1}{3}) = (-4)(-4) - (-2)^2 = 12
\]

So rel min \( \frac{1}{3}, \frac{1}{3} \), since \( P_{pp}(\frac{1}{3}, \frac{1}{3}) < 0 \), has a rel. max.
If asked for the largest possible value: 

(use $r = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$).

\[ P \left( \frac{1}{3}, \frac{1}{3} \right) = 2 \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) - 2 \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) + 2 \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) = \frac{0}{3}. \]