Guide for Section 5.2: Integration by Substitution

Ex 7 A corporation’s profit, generated from advertising deals, is deposited continuously at a rate of 1.8 million dollars per year into an account (initially containing no money) which earns interest at an annual rate of 5%, compounded continuously.

a) Write a differential equation to model the change in value of the account over time.

Let $E(t)$ be the amount of money in the account in millions of $ at time $t$ years with $t=0$ being the starting time. The initial condition is $E(0) = 0$. For the differential eqn, there are two contributions to $E'(t)$, namely, deposit of money: $1.8$ (million of $ per year); and the interest: $0.05 E(t)$. So eqn is $E'(t) = 1.8 + 0.05 E(t)$.

b) Solve the initial value problem from part (7a).

\[
E(t) = 1.8 + 0.05 E(t)
\]

Separate:

\[
\frac{dE}{dt} = 1.8 + 0.05 E
\]

\[
\int \frac{dE}{1.8 + 0.05 E} = \int dt
\]

On left: Substitute $u = 1.8 + 0.05 E$, $du = 0.05 dE$, so $dE = \frac{1}{0.05} du$.

\[
\int \frac{du}{u} = \frac{1}{0.05} \ln(u) + C_1 = \frac{1}{0.05} \ln(1.8 + 0.05 E) + C_1
\]

Thus:

\[
\frac{1}{0.05} \ln(1.8 + 0.05 E) = t + C_2 - C = t + C
\]

\[
\ln(1.8 + 0.05 E(t)) = 0.05 (t + C)
\]

Solve for $E = E(t)$: $1.8 + 0.05 E(t) = e^{0.05(t + C)}$ with $C = 0.05 C_0$. 

CQ (Section 5.2, #3)
\begin{align*}
\text{Given:} & \quad 0.05 \ E(t) = C e^{0.05t} - 1.8 \\
E(t) &= \frac{C}{0.05} e^{0.05t} - \frac{1.8}{0.05} \\
\text{Since: } E(0) &= 0, \quad \text{we can say} \\
0 &= E(0) = \frac{C}{0.05} e^{0} - \frac{1.8}{0.05} \\
&= \frac{C}{0.05} - \frac{1.8}{0.05} \\
C &= 1.8 \\
\text{Thus, } & \quad E(t) = \frac{1.8}{0.05} e^{0.05t} - \frac{1.8}{0.05} \\
&= 36 \ e^{0.05t} - 36.
\end{align*}
Ex 8 The number \( S \) of shares available for the public to trade (a company’s “float”) at time \( t \) hours after its initial public offering may follow a Gompertz model, which is of the form

\[
\frac{dS}{dt} = -0.05S \cdot \ln \left( \frac{K}{S} \right)
\]

for a positive constant \( K \) which represents the number of shares in total. If there are 3 million shares in total, initially with float 2.8 million shares, find a model for the company’s float as a function of \( t \).

\[
\text{Units for } S: \text{ million of shares} \\
\text{Units for } t: \text{ year}.
\]

\( S = 3 \text{ million} \)

The differential equation:

\[
\frac{dS}{S \cdot \ln(\frac{S}{K})} = -0.05 \, dt
\]

\[
\int \frac{dS}{S \cdot \ln(\frac{S}{K})} = \int 0.05 \, dt = 0.05 \, t + C.
\]

Substitute \( u = \ln(\frac{S}{K}) \).

Note: We see \( du = \frac{dS}{S} \) in the integrand.
\( \frac{du}{ds} = \frac{1}{S} \)

Put in the substitution:

\[
\int \frac{du}{u} = \ln(|u|) + C = \ln(|h(t) - h(k)|) + C
\]

so

\[
\ln \left( |\ln(S) - \ln(K)| \right) = 0.05t + C
\]

When \( t = 0 \), \( S = 2.8 \). Also \( K = 3 \).

\[
\ln \left( |\ln(S) - \ln(K)| \right) = 0.05t + C
\]

Put \( t = 0 \) and \( S = 2.8 \) to get

\[
\ln \left( |\ln(S) - \ln(3)| \right) = C
\]

Note: \( \exp(x) = e^x \)

Since \( S < K = 3 \) so \( \ln(S) - \ln(K) < 0 \)

so

\[
|\ln(S) - \ln(K)| = \ln(K) - \ln(S).
\]

Rewrite:

\[
C = \ln \left( \ln(K) - \ln(S) \right)
\]

\[
\ln \left( |\ln(S) - \ln(3)| \right) = 0.05t + C
\]

\[
\ln(S) - \ln(3) = e^{0.05t + C}, \quad \ln(S) = e^{0.05t + C} + \ln(3)
\]

\[
S = \exp \left( e^{0.05t + C} - \ln(3) \right) = \exp \left( e^{0.05t + C} \right) \exp(\ln(3)) = 3 \exp \left( e^{0.05t + C} \right)
\]
\[ e^{0.05t + C} = e^{0.05t} e^C \]

\[ = e^{0.05t} e^{\ln(\ln(3) - \ln(2.8))} \]

\[ = e^{0.05t} e^{\ln(3) - \ln(2.8)} \]

So

\[ s = 3 \exp(e^{0.05t + C}) \]

\[ = 3 \exp(e^{0.05t} [\ln(3) - \ln(2.8)]) \]