**Ex 1** Find the area between the graph of \( y = 4 - 3x \) and the two coordinate axes.

\[ \text{(Done on Monday.)} \]

**Ex 2** Use four rectangles to approximate the area underneath the graph of \( y = f(x) = 16 - \frac{1}{4}x^2 \) in the first quadrant.

\[ \text{Done on Monday, but let's go} \]
\[ \text{over it again quickly.} \]
\[ \text{To keep things neat, here use} \]
\[ \text{the left endpoints of the intervals.} \]
\[ \text{Approximate is:} \]
\[ 16 \cdot 2 + 15 \cdot 2 + 12 \cdot 2 + 7 \cdot 2 \]
\[ f(0) \quad f(2) \quad f(4) \quad f(6) \]
\[ = (16 + 15 + 12 + 7)(2). \]

**Right endpoints:**
\[ 15 \cdot 2 + 12 \cdot 2 + 7 \cdot 2 + 0 \cdot 2. \]
Guide for Section 5.3: The Definite Integral

Def A left Riemann sum for \( f(x) \) on the interval \([a, b]\) is the area of \( n \) equal width rectangles, with height given by the value of \( f \), which is a computation of the form

\[
[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)] \Delta x,
\]

where \( x_1, x_2, \ldots, x_n \) is a list of the equally spaced left endpoints at which each rectangle of width \( \Delta x = \frac{b-a}{n} \) is placed, so that \( x_1 = a \) and \( x_{n+1} = b \).

(There are more general Riemann sums. See the book.)

Ex 3 Express the computation from Example 2 as a Riemann sum by finding values for \( a, b, n, \Delta x, \) and \( x_1, x_2, \ldots, x_n \).

\[
\begin{align*}
a &= 0 \\
b &= 8 \\
n &= 4 \\
\Delta x &= \frac{b-a}{n} = 2 \\
x_1 &= 0 \\
x_2 &= 2 \\
x_3 &= 4 \\
x_4 &= 6
\end{align*}
\]

(CQ) (Section 5.3, #1)

Def (Definite Integral) Let \( f \) be a function defined on \([a, b]\) which is continuous there (or is bounded and is discontinuous at only finitely many points). The definite integral of \( f \) over the interval \([a, b]\), written \( \int_a^b f(x) \, dx \), is the limit of the Riemann sum as the number of rectangles increases without bound. That is,

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \left[ f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right] \Delta x,
\]

where \( x_1, x_2, \ldots, x_n \) is a list of the left endpoints at which each rectangle of width \( \Delta x \) is placed.

Geometrically, the definite integral represents the area between the graph of \( y = f(x) \), the \( x \)-axis, and the lines \( x = a \) and \( x = b \) (with area below the \( x \)-axis considered to be negative).
Guide for Section 5.3: The Definite Integral

(Definite integrals can be defined more generally. We won’t see any thing like this in this course, and it doesn’t normally happen for functions seen in applications, but the limit above may exist even when the definite integral, as on page 410 of the book, does not exist.)

**Ex 4** Express the computation from Example 1 as a definite integral.

$$\int_{0}^{3} (4-3x) \, dx$$

These must match.

**Ex 5** Express the computation from Example 2 as an approximation of a definite integral.

$$\int_{0}^{8} f(x) \, dx = \int_{0}^{8} (16-\frac{4}{3}x^2) \, dx$$

$$\int_{0}^{8} f(x) \, dx = \int_{0}^{8} (16+15+2+7) \, dx$$

$$\int_{0}^{8} f(x) \, dx = \int_{0}^{8} (24) \, dx$$

**Ex 6** Let $A$ be the area between $x = -3$, $x = 0$, and the graph of $y = f(x)$. Let $B$ be the area between $x = 0$, $x = 4$, and the graph of $y = f(x)$. Let $C$ be the area between $x = 4$, $x = 6$, and the graph of $y = f(x)$. Find an expression for $\int_{-3}^{6} f(x) \, dx$ in terms of $A$, $B$, and $C$.

$$\int_{-3}^{6} f(x) \, dx = A-B+C$$

See sign amendment on previous page.

**Ex 7** Let $f(t)$ be given by the complete graph shown. Evaluate $\int_{-4}^{4} f(t) \, dt$.

$$\int_{-4}^{4} f(t) \, dt = \int_{-4}^{2} f(t) \, dt + \int_{2}^{0} f(t) \, dt + \int_{0}^{-2} f(t) \, dt$$

$$+ \int_{-2}^{-4} f(t) \, dt + \int_{-4}^{2} f(t) \, dt$$

$$= A-B-C-D+E = 1-1-2-\frac{1}{2}$$

**CQ** (Section 5.3, #2)

This expression is a preview of the rule on page 5?

Area of triangle with height 1 and base 2

Area of rectangle with base 2, height 1
Guide for Section 5.3: The Definite Integral

**Thm** (Part of the Fundamental Theorem of Calculus) For any function \( f \) continuous on the interval \([a, b]\), and any antiderivative \( F \) of \( f \), we have

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

*Important part of full theorem: If \( f \) is continuous, then there is an antiderivative \( F \) of \( f \), even if you can't write it.*

**Ex 8** Compute \( \int_0^8 (16 - \frac{1}{4}x^2) \, dx \). Compare with the approximation from Example 2.

\[
f(x) = 16 - \frac{1}{4}x^2.
\]

Take \( F(x) = 16x - \frac{1}{12}x^3 \)

Then find \( F(8) - F(0) \)

\[
16 \cdot 8 - \frac{1}{12} \cdot 8^3 = 2 \cdot 8^2 \cdot \frac{4}{3} \cdot 8 = \frac{256}{3}
\]

\[
\approx 85.333
\]

*Don't use much fractions.*

From Ex 2, \( (16 + 15 + 12 + 7) \cdot 2 = 100 \).

From right endpoint: \( (15 + 12 + 7 + 0) \cdot 2 = 74 \)
Standard way to write this:

$$\int_{0}^{f} f(x) \, dx = \int_{0}^{f} \left(16 - \frac{1}{4} x^2\right) \, dx = \left[16x - \frac{1}{12} x^3\right]_{0}^{f}$$

Not evaluated

$$= 16f - \frac{1}{12} (f^3) - \left(16 \cdot 0 - \frac{1}{12} (0^3)\right)$$

$$= \frac{256}{3}.$$

The symbol $F(b) \bigg|_{a}^{b}$ means $F(b) - F(a)$.

If have the name $F$, this is pointless.

But here we only have the formula, without a name for the function.
Guide for Section 5.3: The Definite Integral

Wait, why don’t we need “+C”? Try using the substitution \( G(x) = 16x - \frac{1}{12}x^2 + 4 \).

\[
\int_0^4 f(x) \, dx \quad \text{shall be} \quad G(4) - G(0).
\]

\[
G(4) = 16 \cdot 4 - \frac{1}{12} (4^2) + 4 = 64 - \frac{16}{3} + 4 = \frac{180}{3} - \frac{16}{3} = \frac{164}{3},
\]

\[
G(0) = 16 \cdot 0 - \frac{1}{12} (0^2) + 4 = 4.
\]

\[
\int_0^4 f(x) \, dx = \frac{164}{3} - 4 = \frac{164 - 12}{3} = \frac{152}{3}.
\]

**Thm** (Rules for Definite Integrals) The same constant multiple, sum, and difference rules that we know for indefinite integrals also apply to definite integrals. Additionally, if \( a < b < c \), then

\[
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.
\]

**Ex 9** Suppose that for functions \( f(x) \) and \( g(x) \), compute each indicated definite integral, given that

\[
\int_{-2}^1 f(x) \, dx = 5, \quad \int_{-2}^1 g(x) \, dx = -2, \quad \int_1^3 f(x) \, dx = -1, \quad \text{and} \quad \int_{-2}^0 f(x) \, dx = 3.
\]

a) \[
\int_{-2}^1 [2f(x) - g(x)] \, dx = 2 \int_{-2}^1 f(x) \, dx - \int_{-2}^1 g(x) \, dx = 2(5) - (-2) = 12.
\]

b) \[
\int_{-2}^3 f(x) \, dx = \int_{-2}^1 f(x) \, dx + \int_1^3 f(x) \, dx = 5 + (-1) = 4.
\]
c) \( \int_0^1 f(x) \, dx \)

\[
\int_{-2}^1 f(x) \, dx = \int_{-2}^0 f(x) \, dx + \int_0^1 f(x) \, dx.
\]

\[
\int_0^1 f(x) \, dx = 3 + \int_0^1 f(x) \, dx.
\]

\[
\int_0^1 f(x) \, dx = 2.
\]

\( \text{CQ (Section 5.3, #3)} \)

\( \text{Thm} \) Let \( f \) be \( \mathcal{C} \) on the interval \([a, b]\) such that \( f' \) exists and is continuous on \([a, b]\). The net change of \( f \) over the interval \([a, b]\) is

\[
f(b) - f(a) = \int_a^b f'(t) \, dt.
\]

(This also works for reasonable continuous piecewise defined functions, for which the derivative may fail to exist at finitely many points, such as the function graphed in Example 7.)