Ex 1 Find the area between the graph of \( y = 4 - 3x \) and the two coordinate axes.

Ex 2 Use four rectangles to approximate the area underneath the graph of \( y = f(x) = 16 - \frac{1}{4}x^2 \) in the first quadrant.
**Guide for Section 5.3: The Definite Integral**

**Def** A left Riemann sum for $f(x)$ on the interval $[a, b]$ is the area of $n$ equal width rectangles, with height given by the value of $f$, which is a computation of the form

$$[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)] \Delta x,$$

where $x_1, x_2, \ldots, x_n$ is a list of the equally spaced left endpoints at which each rectangle of width $\Delta x = \frac{b-a}{n}$ is placed, so that $x_1 = a$ and $x_{n+1} = b$.

(There are more general Riemann sums. See the book.)

**Ex 3** Express the computation from Example 2 as a Riemann sum by finding values for $a$, $b$, $n$, $\Delta x$, and $x_1, x_2, \ldots, x_n$.

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**CQ** (Section 5.3, #1)

**Def** (Definite Integral) Let $f$ be a function defined on $[a, b]$ which is continuous there (or is bounded and is discontinuous at only finitely many points). The definite integral of $f$ over the interval $[a, b]$, written $\int_a^b f(x) \, dx$, is the limit of the Riemann sum as the number of rectangles increases without bound. That is,

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \left[ f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right] \Delta x,$$

where $x_1, x_2, \ldots, x_n$ is a list of the left endpoints at which each rectangle of width $\Delta x$ is placed.

Geometrically, the definite integral represents the area between the graph of $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$ (with area below the $x$-axis considered to be negative).
(Definite integrals can be defined more generally. We won’t see any thing like this in this course, and it doesn’t normally happen for functions seen in applications, but the limit above may exist even when the definite integral, as on page 410 of the book, does not exist.)

**Ex 4** Express the computation from Example [1] as a definite integral.

**Ex 5** Express the computation from Example [2] as an approximation of a definite integral.

**Ex 6** Let $A$ be the area between $x = -3, x = 0,$ and the graph of $y = f(x)$. Let $B$ be the area between $x = 0, x = 4,$ and the graph of $y = f(x)$. Let $C$ be the area between $x = 4, x = 6,$ and the graph of $y = f(x)$. Find an expression for $\int_{-3}^{6} f(x) \, dx$ in terms of $A$, $B$, and $C$.

**Ex 7** Let $f(t)$ be given by the complete graph shown. Evaluate $\int_{-4}^{4} f(t) \, dt$.

[CQ](Section 5.3, #2)
Thm (Part of the Fundamental Theorem of Calculus) For any function $f$ continuous on the interval $[a, b]$, and any antiderivative $F$ of $f$, we have

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Ex 8 Compute $\int_0^8 \left(16 - \frac{1}{4}x^2\right) \, dx$. Compare with the approximation from Example 2.
Wait, why don’t we need “+C”?

[Thm] (Rules for Definite Integrals) The same constant multiple, sum, and difference rules that we know for indefinite integrals also apply to definite integrals. Additionally, if $a < b < c$, then

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.$$  

Ex 9 Suppose that for functions $f(x)$ and $g(x)$, compute each indicated definite integral, given that

$$\int_{-2}^1 f(x) \, dx = 5, \quad \int_{-2}^1 g(x) \, dx = -2, \quad \int_1^3 f(x) \, dx = -1, \quad \text{and} \quad \int_{-2}^0 f(x) \, dx = 3.$$

a) $\int_{-2}^1 [2f(x) - g(x)] \, dx$

b) $\int_{-2}^3 f(x) \, dx$
c) \[ \int_{0}^{1} f(x) \, dx \]

**Thm** Let \( f \) be a function on the interval \([a, b]\) such that \( f' \) exists and is continuous on \([a, b]\). The net change of \( f \) over the interval \([a, b]\) is

\[
f(b) - f(a) = \int_{a}^{b} f'(t) \, dt.
\]

(This also works for reasonable continuous piecewise defined functions, for which the derivative may fail to exist at finitely many points, such as the function graphed in Example 7.)
Ex 10 The rate of change of the accumulated profit of a growing business is given by the graph below (from Example 6), with \( x \) measured in years from 2000 and \( y \) measured in millions of dollars per year.

\[ y = f(x) \]

a) Write the net change in total profit between years 2000 and 2006 as a definite integral.

b) Is the net change in total profit between 2000 and 2006 positive or negative?

c) What are the units of net profit?
Ex 11 A company’s monthly production rate changes each month by \(\frac{4}{3t + 1}\) thousand items per month \(t\) months after the product’s public release. (The units of the rate of change are “thousands of items/month\(^2\”).) Find the change in monthly production between three months and six months after release. Include units.

CQ (Section 5.3, #4)

Ex 12 Compute \(\int_{-1}^{2} \frac{e^{3t} + e^{-3t}}{e^{3t}} \, dt\).
Ex 13 Suppose $R(t)$ is the rate of change of the national debt of the Wreozian Empire, in billions of dollars per year, with $t$ measured in years. What are the units of $\int R(t) \, dt$?

Ex 14 Social networking platform Chirper’s market share $M$ decreases at the rate of $\frac{dM}{dx} = \frac{-x}{\sqrt{x^2 + 100}}$ percentage points per additional point of competitor Expregram’s market share, where $x$ is Expregram’s current market share (as a percentage). In the course of a year, Expregram’s market share grows from 20% to 28%. What is the net change in market share for Chirper?