STOP

I strongly recommend that you not review these solutions until you have (a) studied first and (b) gotten as far as you could with these problems by yourself. Just looking at the answers is not the same as doing them on your own.
1. Find a family of antiderivatives using the power rule, the antiderivative of \( x^{-1} \), and the rule for exponential functions.

Example: Find an expression for all antiderivatives of \( Q(n) = 9n^{-1} - 7n^{-2} \).

All antiderivatives can be found as the indefinite integral of \( Q \), so

\[
\int Q(n) \, dn = \int (9n^{-1} - 7n^{-2}) \, dn
\]

\[
= 9 \cdot \ln(|n|) - 7 \cdot n^{-2+1} + C
\]

\[
= 9 \ln(|n|) + 7n^{-1} + C
\]

2. Use sum, difference, and constant multiple rules for indefinite and definite integrals.

Example: The government of a certain country estimates that \( t \) years after the beginning of 2015, imports will be increasing at the rate \( I'(t) \) and exports at the rate \( E'(t) \), both in (billions of dollars per year)/year, where

\[
I'(t) = 12.5e^{0.2t} \quad \text{and} \quad E'(t) = 1.7t + 3.
\]

The trade deficit is the difference between imports and exports in a given year. Write a formula for the trade deficit \( D(t) \) assuming that at the beginning of 2016 the trade deficit was 7 billion dollars per year.

We are told that the trade deficit is the difference in imports and exports, so \( D(t) = I(t) - E(t) \). We can find formulas for \( I(t) \) and \( E(t) \) by integrating each expression, and then the difference is the trade deficit formula. First the import function:

\[
\int 12.5e^{0.2t} \, dt = 12.5 \cdot \frac{1}{0.2} e^{0.2t} + C_1 = 62.5e^{0.2t} + C_1.
\]

Then the export function:

\[
\int (1.7t + 3) \, dt = 1.7 \cdot \frac{t^2}{2} + 3t + C_2 = 0.85t^2 + 3t + C_2.
\]

The general solution for the trade deficit then is

\[
D(t) = (62.5e^{0.2t} + C_1) - (0.85t^2 + 3t + C_2)
\]

\[
= 62.5e^{0.2t} - 0.85t^2 - 3t + C.
\]

The particular solution can be found using the fact that in 2016 the trade deficit was 7 billion dollars, so \( D(1) = 7 \):

\[
7 = 62.5e^{0.2(1)} - 0.85(1)^2 - 3(1) + C
\]

\[
C \approx -65.49
\]

Thus the trade deficit function is

\[
D(t) = 62.5e^{0.2t} - 0.85t^2 - 3t - 65.49
\]
3. Evaluate definite integrals involving the property that for $a < c < b$, we have

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$  

Example: Suppose that for a continuous function $g(t)$, we know that $\int_{-1}^2 g(t) \, dt = 3$, while $\int_{1}^2 g(t) \, dt = -1$. Find the value of $\int_{-1}^1 g(t) \, dt$.

From the properties of definite integrals, we know that

$$\int_{-1}^2 g(t) \, dt = \int_{-1}^1 g(t) \, dt + \int_1^2 g(t) \, dt.$$  

We know two of these values, so that

$$3 = \int_{-1}^1 g(t) \, dt + (-1),$$

so we can solve for the unknown definite integral:

$$\int_{-1}^1 g(t) \, dt = 4.$$  

Example: A company wants to measure the effect of a viral marketing campaign. The company believes that the rate of change in the fraction of the population that has heard of the item is proportional to the difference between the entire population and the fraction of people who have heard of it. Set up a differential equation to model this rate of change.

Let $f(t)$ be the fraction of the population that has heard of the item at time $t$ (with $t$ measured in, say, days). Translating bit by bit, “the rate of change in the fraction of the population that has heard of the item” translates to

$$f'(t),$$

the phrase “is proportional to” translates to

$$= k \cdot ( \ )$$

for some constant $k$ and some as yet undetermined expression inside the parentheses, and the phrase “the difference between the entire population and how many people have heard of it” translates to

$$1 - f(t),$$

since the fraction representing the entire population is just 100%, or 1. Thus the differential equation is

$$f'(t) = k(1 - f(t)) \text{ for some constant } k.$$
5. Verify a provided solution to a differential equation.

First we compute (using product rule)

\[ y'(t) = \frac{d}{dt} [2e^t + te^t] = 2e^t + e^t + te^t = 3e^t + te^t. \]

Next we get

\[ y''(t) = \frac{d}{dt} [3e^t + te^t] = 3e^t + e^t + te^t = 4e^t + te^t. \]

Then we put all this together into the differential equation to see if both sides are equal:

\[
(4e^t + te^t) - 2(3e^t + te^t) + (2e^t + te^t) = 0
\]

\[
4e^t - 6e^t + 2e^t + te^t - 2te^t + te^t = 0 \checkmark
\]

So we have confirmed that the given solution works for the differential equation.

6. Find a general or particular solution to a simple differential equation by integration.

We can solve the differential equation with integration to find a general solution:

\[ H'(t) = te^{-t^2} \]

\[ H(t) = \int te^{-t^2} dt. \]

We can find the value of the integral on the right hand side using the substitution \( u = -t^2 \), so that \( du = -2t \, dt \), or \( -\frac{1}{2} \, du = t \, dt \). Thus

\[
\int te^{-t^2} dt = \int e^u \left( -\frac{1}{2} \right) du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-t^2} + C.
\]

So \( H(t) = -\frac{1}{2} e^{-t^2} + C \).

We can find the particular solution using the fact that \( H(0) = 25 \), so that

\[
H(t) = -\frac{1}{2} e^{-t^2} + C
\]

\[ 25 = H(0) = -\frac{1}{2} e^{0^2} + C \]

\[ 25 = -\frac{1}{2} (1) + C \]

\[ C = 25.5 \]
Thus, our particular solution is

\[ H(t) = -\frac{1}{2}e^{-t^2} + 25.5. \]

7. Find a general or particular solution to a separable differential equation by integration.

We solve by separating variables:

\[ \frac{dy}{dx} = 10 - 4y \]
\[ dy = (10 - 4y) \, dx \]
\[ \frac{1}{10 - 4y} \, dy = dx \]
\[ \int \frac{1}{10 - 4y} \, dy = \int dx \]

We can use the substitution \( u = 10 - 4y \), so that \( du = -4 \, dy \), or \(-\frac{1}{4}du = dy\). Then we have

\[ \int \frac{1}{10 - 4y} \, dy = \int dx \]
\[ \int \frac{1}{u} \left(-\frac{1}{4}\right) \, du = \int dx \]
\[ -\frac{1}{4} \ln(|u|) = x + C_1 \]
\[ \ln(|10 - 4y|) = -4x + C \quad \text{with} \quad C = -4C_1 \]
\[ |10 - 4y| = e^{-4x+C} \]

We know that \( y > 3 \), so then \( 4y > 12 \), and \( 10 - 4y < -2 \). We can remove the absolute values by replacing \( |10 - 4y| \) by \(-(10 - 4y)\), since we are guaranteed that \(10 - 4y\) is negative.

\[ |10 - 4y| = e^{-4x+C} \]
\[ -(10 - 4y) = e^{-4x+C} \]
\[ 4y - 10 = e^{-4x+C} \]
\[ 4y = e^{-4x+C} + 10 \]
\[ y = \frac{1}{4}e^{-4x+C} + 2.5 \]

8. Find a family of antiderivatives using substitution.

**Example:** Find the general solution to the differential equation \( \frac{dy}{dx} = 10 - 4y \), assuming \( y > 3 \).

\[ \frac{dy}{dx} = 10 - 4y \]
\[ dy = (10 - 4y) \, dx \]
\[ \int \frac{1}{10 - 4y} \, dy = \int dx \]
\[ \int \frac{1}{u} \left(-\frac{1}{4}\right) \, du = \int dx \]
\[ -\frac{1}{4} \ln(|u|) = x + C_1 \]
\[ \ln(|10 - 4y|) = -4x + C \quad \text{with} \quad C = -4C_1 \]
\[ |10 - 4y| = e^{-4x+C} \]

We know that \( y > 3 \), so then \( 4y > 12 \), and \( 10 - 4y < -2 \). We can remove the absolute values by replacing \( |10 - 4y| \) by \(-(10 - 4y)\), since we are guaranteed that \(10 - 4y\) is negative.

\[ |10 - 4y| = e^{-4x+C} \]
\[ -(10 - 4y) = e^{-4x+C} \]
\[ 4y - 10 = e^{-4x+C} \]
\[ 4y = e^{-4x+C} + 10 \]
\[ y = \frac{1}{4}e^{-4x+C} + 2.5 \]

**Example:** Find \( \int \frac{12x - 8}{\sqrt{3x^2 - 4x + 16}} \, dx \).
We may use the substitution \( u = 3x^2 - 4x + 16 \), so that \( \frac{du}{dx} = 6x - 4 \), or \( \frac{1}{6x - 4} \, du = dx \). Then

\[
\int \frac{12x - 8}{\sqrt{3x^2 - 4x + 16}} \, dx = \int \frac{12x - 8}{\sqrt{u}} \left( \frac{1}{6x - 4} \right) \, du = \int \frac{2(6x - 4)}{\sqrt{u}} \left( \frac{1}{6x - 4} \right) \, du = \int 2u^{-1/2} \, du = 2u^{1/2} + C = 4(3x^2 - 4x + 16)^{1/2} + C.
\]

Note: This substitution only works because 12x – 8 is a constant multiple of \( \frac{d}{dx} (3x^2 - 4x + 16) \).

Example: The graph of \( y = f(t) \) is provided. Find the value of \( \int_{-4}^{3} f(t) \, dt \).

The area of the rectangle and small triangle underneath the \( x \)-axis on the left side of the graph (but to the right of \( x = -4 \)) is \( A_1 = (1) \cdot 2 + \frac{1}{2} (0.5)(1) = 2.25 \), while the area above the \( x \)-axis is the triangle with area \( A_2 = \frac{1}{2} (4.5)(3) = 6.75 \). Thus we get

\[
\int_{-4}^{3} f(t) \, dt = A_2 - A_1 = 6.75 - 2.25 = 4.5
\]

Example: Approximate the value of \( \int_{1}^{16} 3t \ln(t) \, dt \) with a left Riemann sum using 5 intervals of equal length.

9. Interpret a definite integral as the (signed) area between a curve and the horizontal axis.

10. Compute an indicated Riemann sum.
We see $a = 1$, $b = 16$, so that $\Delta t = \frac{16 - 1}{5} = 3$, and let’s define $f(t) = 3t \ln(t)$. We find $t_0 = 1$, $t_1 = 4$, $t_2 = 7$, $t_3 = 10$, and $t_4 = 13$. Then our Riemann sum is

\[
[f(t_0) + f(t_1) + f(t_2) + f(t_3) + f(t_4)] \cdot \Delta t \\
= [f(1) + f(4) + f(7) + f(10) + f(13)] \cdot 3 \\
= [3(1) \ln(1) + 3(4) \ln(4) + 3(7) \ln(7) + 3(10) \ln(10) + 3(13) \ln(13)] \cdot 3 \\
\approx 679.83
\]

11. Find the value of a definite integral using the Fundamental Theorem of Calculus.

Example: The promoters of a county fair estimate that $t$ hours after the gates open at 9:00 am, visitors will be entering the fair at the rate of $-4(t + 2)^3 + 54(t + 2)^2$ people per hour. How many people will enter the fair between 10:00 am and noon?

We can interpret this as a definite integral of the rate, from $t = 1$ to $t = 3$. The number of fair-goers is given by

\[
\int_1^3 [-4(t + 2)^3 + 54(t + 2)^2] \, dt.
\]

We do the indefinite integral using the substitution $u = t + 2$, so that $du = dt$:

\[
\int (-4u^3 + 54u^2) \, du = \left[ -4 \cdot \frac{u^4}{4} + 54 \cdot \frac{u^3}{3} \right] + C \\
= -(t + 2)^4 + 18(t + 2)^3 + C.
\]

Therefore

\[
\int_1^3 [-4(t + 2)^3 + 54(t + 2)^2] \, dt = \left[ -(t + 2)^4 + 18(t + 2)^3 \right]_1^3 \\
= \left[ -(3 + 2)^4 + 18(3 + 2)^3 \right] - \left[ -(1 + 2)^4 + 18(1 + 2)^3 \right] \\
= 1220.
\]

Thus 1220 people will enter the fair between 10:00 am and noon.

12. Interpret the definite integral of a marginal function as a net change in the original function.

Example: A farmer’s crop yield will be increasing at the rate of $0.5t^2 + 4(t + 1)^{-1}$ bushels per day, $t$ days from now. By how much will the value of the crop increase during the next 6 days if the market price remains fixed at $\$2$ per bushel?

The value of the crop at time $t$ can be expressed as $V(t) = p \cdot q(t)$, where $p$ is the price per bushel and $q(t)$ is the number of bushels produced. Because the price is held constant, the net change in value, is then

\[
\Delta V = \int_0^6 p \cdot q'(t) \, dt = \int_0^6 2 \cdot (0.5t^2 + 4(t + 1)^{-1}) \, dt = \int_0^6 \left[ t^2 + 8(t + 1)^{-1} \right] \, dt.
\]
We can use the substitution \( u = t + 1 \) for the second term in the integrand:

\[
\int [t^2 + 8(t + 1)^{-1}] \, dt = \int t^2 + \int 8u^{-1} \, du \\
= \frac{t^3}{3} + 8 \ln(|u|) + C \\
= \frac{t^3}{3} + 8 \ln(|t + 1|) + C.
\]

So

\[
\Delta V = \int_{0}^{6} [t^2 + 8(t + 1)^{-1}] \, dt \\
= \left[ \frac{t^3}{3} + 8 \ln(|t + 1|) \right]_{0}^{6} \\
= \left[ \frac{t^3}{3} \right]_{0}^{6} + \left[ 8 \ln(|t + 1|) \right]_{0}^{6} \\
= \frac{6^3}{3} - \frac{0^3}{3} + 8 \ln(|6 + 1|) - 8 \ln(|0 + 1|) \\
\approx 87.67 \text{ dollars}
\]
Additional Free Response-Style Questions

13. Let \( W(x) \) be a continuous function such that \( \int_{1}^{4} W(x) \, dx = 9 \) and \( \int_{0}^{1} W(x) \, dx = \frac{5}{3} \).

(a) Find the value of \( \int_{0}^{4} [W(x) + 1] \, dx \).

\[
\int_{0}^{4} [W(x) + 1] \, dx = \int_{0}^{4} W(x) \, dx + \int_{0}^{4} 1 \, dx
\]

\[
= \int_{0}^{4} W(x) \, dx + \int_{1}^{4} W(x) \, dx + x \bigg|_{0}^{4}
\]

\[
= \frac{5}{3} + 9 + x \bigg|_{0}^{4}
\]

\[
= \frac{5}{3} + 9 + 4 - 0
\]

\[
= \frac{44}{3}
\]

\[
\approx 14.667
\]

(b) If \( W(x) \) represents the annual workforce at a company (in millions of worker-hours) when the average wage is \( x \) dollars per worker-hour, what are the units of \( \int_{1}^{4} W(x) \, dx \)?

The units of an integral (indefinite or definite) are the product of the units of the integrand function (here \( W(x) \)) and the differential variable (here \( x \)); In this case, that is “millions of worker-hours” multiplied by “dollars per worker-hour”, hence \text{millions of dollars}.

(c) If \( \int W(x) \, dx = 2x^2 - \frac{1}{3}x^3 + C \), then what is the formula for \( W(x) \)?

If the result of an indefinite integral is \( 2x^2 - \frac{1}{3}x^3 + C \), then the integrand must be

\[
W(x) = \frac{d}{dx} \left[ 2x^2 - \frac{1}{3}x^3 + C \right] = 4x - x^2.
\]
14. Over time, the demand for premium gasoline is decreasing, while the demand for regular gasoline rises. Consider the rate of change functions $P(t) = \frac{-2}{(t + 1)^{1.2}}$ million gallons of premium gasoline demanded per year and $R(t) = \frac{9}{(t + 1)^{0.8}}$ million gallons of regular gasoline demanded per year, where $t$ is in years after the beginning of 2010. Which of the two gasoline qualities experienced a larger net change in demand from the beginning of 2015 until the beginning of 2020, the rise in demand for regular gasoline or the fall in demand for premium gasoline?

The net change in demand for premium gasoline is the integral of the rate of change in demand for premium gasoline from $t = 5$ to $t = 10$:

$$\int_5^{10} \frac{-2}{(t + 1)^{1.2}} \, dt = \left[ -2 \cdot \frac{(t + 1)^{-0.2}}{-0.2} \right]_5^{10}$$

$$= \left[ -2 \cdot \frac{(10 + 1)^{-0.2}}{-0.2} \right] - \left[ -2 \cdot \frac{(5 + 1)^{-0.2}}{-0.2} \right]$$

$$\approx -0.798.$$  

The net change in demand for regular gasoline is the integral of the rate of change in demand for regular gasoline from $t = 5$ to $t = 10$:

$$\int_5^{10} \frac{9}{(t + 1)^{0.8}} \, dt = \left[ 9 \cdot \frac{(t + 1)^{0.2}}{0.2} \right]_5^{10}$$

$$= \left[ 9 \cdot \frac{(10 + 1)^{0.2}}{0.2} \right] - \left[ 9 \cdot \frac{(5 + 1)^{0.2}}{0.2} \right]$$

$$\approx 8.299.$$  

The rise in demand for regular gasoline exceeds the decline in the demand for premium gasoline.
15. Marginal cost for a product is $10 per unit and increasing at an instantaneous rate of 2% for each additional unit produced. Marginal revenue is $150 per unit. If all fixed costs total $1,000 (that is, total cost is $1000 when no units are produced), find the profit when 100 units are produced and sold.

We are given that marginal cost is increasing at a constant instantaneous rate, which means it is exponential:

\[ C'(x) = 10e^{0.02x}. \]

Marginal revenue is constant at

\[ R'(x) = 150. \]

We need to find profit, which is given by

\[
P(x) = R(x) - C(x) = \int [R'(x) - C'(x)] \, dx = \int [150 - 10e^{0.02x}] \, dx = 150x - \frac{10}{0.02}e^{0.02x} + K.
\]

We use the fact that a fixed cost of $1000 implies that \( P(0) = -1000 \) (no revenue, plus $1000 in cost) to solve for \( K \):

\[
P(0) = 150(0) - \frac{10}{0.02}e^{0.02(0)} + K = -1000 = -500 + K \\
K = -500.
\]

Then our profit function is \( P(x) = 150x - 500e^{0.02x} - 500 \) and the profit from producing and selling 100 units is

\[
P(100) = 150(100) - 500e^{0.02(100)} - 500 \approx 10805.47 \text{ dollars}.
\]

16. One part of a predator-prey model states that the rate of change in a prey population (with respect to time) is proportional to the product of the prey population \( x \) and some linear function of the predator population \( y \). Set up (but definitely do not try to solve) a differential equation modeling this relationship. Your equation should have three unknown constants in addition to the variable populations \( x \) and \( y \).

The prey population is \( x(t) \), so its rate of change at any given time is \( x'(t) \). Being “proportional to” means this expression “\( = k \cdot ( ) \)” for some constant \( k \) and some yet to be determined expression inside the parentheses. Lastly, a “linear function of \( y \)” can represented as \( my + b \) for constants \( m \) and \( b \). Then the differential equation is

\[
x'(t) = k \cdot x(t) \cdot (my(t) + b) \quad \text{for constants} \ k, \ m, \ \text{and} \ b.
\]
17. Total sales of a product over time may follow a model like the differential equation
\[ \frac{dx}{dt} = k \left( 1 - \frac{x}{S} \right) , \]
where \( x \) is the sales after \( t \) months, \( S \) is the point of market saturation (maximum possible sales), and \( k \) is a constant of proportionality. Solve the differential equation assuming that initially none of the product has been sold. Your answer will contain the constants \( k \) and \( S \).

We solve by separating variables and integrating using the substitution \( u = 1 - \frac{x}{S} \):

\[
\frac{1}{1 - \frac{x}{S}} \, dx = k \, dt
\]

\[
\int \frac{1}{1 - \frac{x}{S}} \, dx = \int k \, dt
\]

\[
\int \frac{1}{u} (-S) \, du = \int k \, dt
\]

\[-S \ln(|u|) = kt + C
\]

\[-S \ln \left( \left| 1 - \frac{0}{S} \right| \right) = kt + C.
\]

We can now solve for the constant knowing that \( t = 0 \) implies \( x = 0 \):

\[-S \ln \left( \left| 1 - \frac{0}{S} \right| \right) = k(0) + C
\]

\[-S \ln(1) = C
\]

\[C = 0.
\]

Now we can finish solving for \( x \):

\[
\ln \left( 1 - \frac{x}{S} \right) = -\frac{1}{S} kt = -\frac{kt}{S}
\]

\[1 - \frac{x}{S} = e^{-\frac{kt}{S}}
\]

\[-\frac{x}{S} = e^{-\frac{kt}{S}} - 1
\]

\[\frac{x}{S} = -e^{-\frac{kt}{S}} + 1
\]

\[x = -Se^{-\frac{kt}{S}} + S.
\]