Guide for Section 5.4: Applications of the Definite Integral

**CQ (Section 5.4, #1)**

Dear last name,

**Ex 4** The net excess profit of an investment plan over another is given by

$$\text{NEP} = \int_a^b [P'_1(x) - P'_2(x)] \, dx,$$

where $P'_1(x)$ and $P'_2(x)$ respectively are the rates of profitability of plan 1 and plan 2. The net excess profit gives the total profit gained by plan 1 over plan 2 in a given time interval. Plan 1 is an investment that is currently increasing in value at an instantaneous rate of $500e^{0.01t}$ dollars per day, as compared to plan 2 which is currently increasing in value at an instantaneous rate of $100e^{0.03t}$ dollars per day. Find the net excess profit during the period from now until plan 1 is no longer increasing faster than plan 2.

**Att:**

$$\int_t^b [P'_1(t) - P'_2(t)] \, dt.$$

Take $t = 0$ to be now, $t$ measured in days from now.

In formula: $a = 0$

Need to find $b$. It is obtained by $500e^{0.01b} = 100e^{0.03b}$

At $t = 0$, plan 1 is growing faster than plan 2.

Solve equation: $5e^{0.01b} = e^{0.03b}$

$$5 = e^{0.02b} = e^{0.003b}$$

So $\ln(5) = 0.02b$, and $b = 50\ln(5)$.

Now integrate:

$$\int_0^{50\ln(5)} [500e^{0.01t} - 100e^{0.03t}] \, dt$$

$$= [500 \cdot \frac{e^{0.01t}}{0.01} - 100 \cdot \frac{e^{0.03t}}{0.03}]_0^{50\ln(5)}$$

$$= [500 \cdot e^{0.01(50\ln(5))} - 100 \cdot e^{0.03(50\ln(5))}] - [500 \cdot 1 - 100 \cdot 1]$$

$$= 21$$
\[ 50 \ln(5) \]

\[ 0.01 \]

\[ e^{0.01 \ln(5) \cdot 50} \]

\[ 0.03 \]

\[ e^{0.03 \ln(5) \cdot 50} \]

\[ 500 \]

\[ e^{0.1} \cdot 1 \]

\[ 100 \]

\[ e^{0.3} \cdot 1 \]

\[ 50,000 \left( e^{0.5 \ln(5)} - 1 \right) - \frac{50,000}{3} \left( e^{1.5 \ln(5)} - 1 \right) \]

(Use a calculator for this.)
Guide for Section 5.4: Applications of the Definite Integral

**Def** The average value of a continuous function $f(x)$ on an interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x) \, dx.$$  

**Ex 5** The quarterly revenue for AWS (Amazon Web Service, the cloud service platform produced by Amazon, and the host of many websites advertised in spam) over the past several years can be predicted roughly by the regression model

$$Q(t) = 32.36t^2 + 11.6t + 50$$

million dollars per quarter, $t$ years after the beginning of 2009. What was the average quarterly revenue for AWS between the beginning of years 2012 and 2015 according to this model?

$$\frac{1}{6-3} \int_3^6 Q(t+1)t \, dt = \frac{1}{3} \int_3^6 \left(32.36 t^2 + 11.6t + 50\right) dt$$

**Note:** $t$ = years, $nt$ = quarters

$$= \frac{1}{3} \left[ \frac{32.36}{3} t^3 + \frac{11.6}{2} t^2 + 50t \right]_3^6$$

$$= \frac{1}{3} \left( \frac{32.36}{3} \cdot 6^3 + \frac{11.6}{2} \cdot 6^2 + 50 \cdot 6 \right) - \frac{32.36}{3} \cdot 3^3$$

$$- \frac{11.6}{2} \cdot 3^2 - 50 \cdot 3$$

Can use calculator.

**CQ** (Section 5.4, #2)
Average value: \[ \frac{4 + 6 + 8 + 8}{4} = 2 \]

\[ \int_{0}^{4} f(x) \, dx \]

\[ 4 - 0 \]
Ex 6 For what value of \( x \) is \( f(x) = \frac{4}{x} - x \) equal to its average value on the interval \([1, 5]\)?

Find the average value.

\[
\frac{1}{5-1} \int_1^5 f(x) \, dx = \frac{1}{4} \int_1^5 \left[ \frac{4}{x} - x \right] \, dx = \frac{1}{4} \left[ 4 \ln(x) - \frac{x^2}{2} \right]_1^5
\]

\[
= \frac{1}{4} \left[ 4 \ln(5) - \frac{25}{2} - 4 \ln(1) + \frac{1^2}{2} \right] = \frac{1}{4} \left( 4 \ln(5) - \frac{23}{2} \right)
\]

\[
= \ln(5) - \frac{23}{8}.
\]

Need to solve: \( f(x) = \ln(5) - 3 \), but is,

\[
\frac{4}{x} - x = \ln(5) - 3,
\]

so \( 4 - x^2 = (\ln(5) - 3)x \), or \( 0 = x^2 - (\ln(5) - 3)x - 4 \).

\[
x = \frac{(\ln(5) - 3) \pm \sqrt{(\ln(5) - 3)^2 + 4 \cdot 4}}{2}
\]

We get:

\[
\text{Def} \quad \text{The Lorenz curve for a group is a function } L(x) \text{ which gives the fraction of income } L(x) \text{ realized by the poorest fraction } x \text{ of the group.}
\]

Ex 7 Consider the Lorenz curve \( L(x) = 0.5x - \frac{1}{x - 2} - 0.5 \).

a) Compute \( L(0) \) and \( L(1) \).

\[
L(0) = 0.5(0) - \frac{1}{0 - 2} - 0.5 = 0
\]

\[
L(1) = 0.5(1) - \frac{1}{1 - 2} - 0.5 = \frac{1}{2} + 1 - \frac{3}{2} = 1
\]

b) What fraction of the income of this group is possessed by the poorest quarter of the group?

\[
L\left(\frac{1}{4}\right) = 0.5\left(\frac{1}{4}\right) - \frac{1}{\frac{1}{4} - 2} - 0.5 = \frac{1}{8} + \frac{4}{7} - \frac{1}{2} = \frac{3}{8} + \frac{4}{7}
\]

\[
= \frac{21 + 32}{56} = \frac{53}{56}.
\]
Guide for Section 5.4: Applications of the Definite Integral

**Def** The income Gini Index is a measure of income inequality within a group. A Gini Index equal to 0 indicates perfectly equal distribution of wealth (that is, everyone has the exact same amount of wealth), while a Gini Index equal to 1 indicates perfectly *unequal* distribution of wealth (that is, one individual controls all of the wealth). The Gini Index for a group can be computed by

\[ GI = 2 \int_0^1 [x - L(x)] \, dx, \]

where \( L(x) \) is the Lorenz curve for the group.

**Ex 8** Find the Gini Index for the United States, with \( L(x) = 1.41x^3 - 0.84x^2 + 0.43x \). By this metric, does the United States have more or less income equality than Moldova, which the CIA considers to have a Gini Index of 0.268 (2014 estimate)?

(Estimated GDP per capita of Moldova in 2017 was $6,700 in 2017 dollars (rank: 143 in the world), less than that of India and that of the Philippines.)

\[
2 \int_0^1 \left[ x - \left( 1.41x^3 - 0.84x^2 + 0.43x \right) \right] \, dx = 2 \int_0^1 \left( -0.57x - 1.41x^2 + 0.84x^2 \right) \, dx
\]

\[
= 2 \left[ -0.57 \frac{x^2}{2} - 1.41 \frac{x^4}{4} + 0.84 \frac{x^3}{3} \right]_0^1 = 2 \left( -0.57 \frac{1}{2} - 1.41 \frac{1}{4} + 0.84 \frac{1}{3} \right)
\]

(use a calculator)