**Def** The future value of a continuous investment (or the future value of a “continuous income stream”) is the total value, including continuously accruing interest, of the income stream at the end of the given time frame. If the income stream is \( f(t) \) (in units of the form “money per year”), invested for \( T \) years at annual interest rate \( r \) (compounded continuously), then its future value is given by

\[
FV = e^{rT} \int_0^T f(t)e^{-rt} \, dt.
\]

Note the assumption: all income from the investment is immediately put into an account paying interest at a fixed rate \( r \).

**Where did this formula come from?**

\[
\int_0^T e^{r(T-t)}f(t) \, dt = \int_0^T e^{r(T-t)}f(t) \, dt = e^{rT} \int_0^T e^{-rt}f(t) \, dt.
\]

**Ex 3** Beginning in the year 2010 and until 2040, GiantCo invests 10 million dollars per year (continuously) in treasury bonds that mature in 2040 and which pay 3.75% interest annually (compounded continuously). Find the future value of GiantCo's investment (that is: the future value of the continuous income stream).

Let \( f(t) \) be the rate at hand \( t \), in millions of $ per year.

Then \( f(t) = 10 \) for all \( t \), and \( r = 0.0375 \). So we get:

\[
T = 30, \text{ and GiantCo invested for } 30 \text{ years since 2010.}
\]

We get:

\[
\int_0^{30} e^{0.0375} \left( 10 e^{-0.0375 t} \right) \, dt
\]

\[
= e^{0.0375 \cdot 30} \left( 10 \left( -\frac{1}{0.0375} e^{-0.0375 t} \right) \right)_{0}^{30}
\]

\[
= e^{1.125} \left( -\frac{1}{0.0375} \right) \left( e^{-9.75} - 1 \right)
\]

\[
= 4.1
\]
\[
\begin{align*}
&= e^{(0.0375)(30)} (\theta) \left[ - \frac{1}{0.0375} \left( e^{-0.0375(30)} \right) \right] + 1 \\
&= \theta e^{(0.0375)(30)} \left( \frac{1}{0.0375} \right) \left( 1 - e^{-(0.0375)(30)} \right) \\
&= \text{(Can be done with a calculator.)} \quad \text{In millions of $}.
\end{align*}
\]
Def The lump sum quantity one must invest at a given interest rate $r$ in order to match the future value of a continuous income stream $f(t)$ is called the income stream’s present value. It is given by $PV = \int_0^T f(t)e^{-rt} \, dt$.

Where did this formula come from? One way to get it: If you have $\int f(t)e^{-rt} \, dt$ now, it will be worth $e^{rt}\int_0^T f(t)e^{-rt} \, dt$, $T$ years from now. The present value of $f(t)dt$ gotten at time $t$ is $e^{-rt}f(t)dt$.

Ex 4 An 18 year old is given a sizable trust fund. Her benefactor chose the quantity to be equivalent to continuously investing an annual income at a rate of $50,000 plus $5,000 per year, with the assumption that both trust fund and income would be accruing interest at a 6% annual rate compounded continuously, and that the two investments would be equal at age 50. How large is the trust fund?

We are computing the present value of this trust, a trust fund with some amount of money, and an income stream. Money will be withdrawn.

Income in the first year is 50
Income in the second year is 55
Income in the third year is 60
Income / year at this time, $t$, in years after beginning of trust fund, is 50 + 5$t$.

We need $\int_0^s (50 + 5t)e^{-0.06t} \, dt$. Go to page 5-1
$$= \int_{0}^{32} 50 e^{-0.06t} \, dt + 5 \int_{0}^{32} t e^{-0.06t} \, dt$$

$$= \frac{50}{0.06} \left[ e^{-0.06t} \right]_{0}^{32} - \frac{50}{0.06} e^{-0.06(32)} + \frac{50}{0.06} \cdot 1$$

$$= \frac{50}{0.06} \left( 1 - e^{-0.06(32)} \right).$$

$$= 5 \left[ \frac{1}{0.06} e^{-0.06t} \left( t - \frac{1}{0.06} \right) \right]_{0}^{32}$$

$$= -\frac{5}{0.06} \left[ e^{-0.06(t + 32)} \right]_{0}^{32}$$

$$= -\frac{5}{0.06} \left[ e^{-0.06(32)} (32 + \frac{1}{0.06}) - (0 + \frac{1}{0.06}) \right]$$

Final answer:

$$\frac{50}{0.06} \left( 1 - e^{-0.06(32)} \right) - \frac{5}{0.06} \left( e^{-0.06(32)} \left( 32 + \frac{0.1}{0.06} \right) - 1 \right)$$

Now use a calculator.
Side note: How to find \( \int te^{kt} \)?

Want a function \( g(t) \) such that \( g'(t) = te^{kt} \).

Guess (naively) \( g(t) = te^{kt} \)

\[ g'(t) = e^{kt} + kte^{kt} \]

Did these steps work? 
To fix fast; \( g_1(t) = \frac{1}{k} e^{kt} \).

\[ g_1'(t) = \frac{1}{k} e^{kt} + te^{kt} \]

Still wrong.

If take \( g(t) - g_1(t) \), with \( h'(t) = \frac{1}{k} e^{kt} \),

\[ h(t) = \frac{1}{k^2} e^{kt} \] will work.

Integration by parts (in Section 6.1, not on our syllabus) is a systematic way of doing what we done above by guessing and checking. It starts with the product rule for differentiation (however, most products can't be integrated using elementary functions).
Ex 5 Advertising on a new video game increases the player base by \( N'(t) = 30e^{-0.06t} \) new players each week, with each player paying a $40 to purchase the game. (Assume that each player purchases exactly one game.) The marginal cost of advertising is a steady $500 per week. What net profit is earned from advertising in the period for which marginal revenue exceeds marginal cost?

\[
N(t) \text{ is the total number of players at time } t.
\]

\[
N'(t) \text{ is the rate at which you get new players.}
\]

\[
N(0) = 1 \text{ # of players}
\]

\[
t \text{ is # weeks, starting with introduction of the game.}
\]

\[
C(t) \text{ total cost.}
\]

Marginal cost is $500/week. \( = C'(t) \)

Marginal revenue is \((40/\text{player}) \cdot (30e^{-0.06t} \text{ players/week}) = 1200e^{-0.06t} \text{ per week, which is the net it shall have.}

\[
R'(t), \text{ with } R(t) \text{ being total revenue.}
\]

Find \( t_0 \) such that \( C(t_0) = R'(t_0) \), i.e.,

\[
500 = 1200e^{-0.06t_0}, \text{ so } -0.06 t_0 = \ln \left( \frac{5}{12} \right),
\]

\[
t_0 = -\frac{1}{0.06} \ln \left( \frac{5}{12} \right) = \frac{1}{0.06} \ln \left( \frac{12}{5} \right).
\]

Now, find \( \int_0^{t_0} [R'(t) - C(t)] dt \)

Go to 6-
\[ = \int_0^t \left[ 1200 \ e^{-0.06t} - 500 \right] dt \]
\[ = \int_0^t 1200 \ e^{-0.06t} \ dt - 500 \ t \]
\[ = \frac{1200}{-0.06} \ e^{-0.06t} \bigg|_0^t - 500 \ t \]
\[ = -\frac{1200}{0.06} \left( e^{-0.06t} - 1 \right) - 500 \ t \]

\[ e^{-0.06 \ t_0} = e^{-\ln\left(\frac{5}{3}\right)} = \frac{5}{3} \]
\[ = -\frac{1200}{0.06} \left( \frac{5}{3} - 1 \right) - 500 \left( \frac{1}{0.06} \right) \ln\left(\frac{12}{5}\right) \]

Use a calculator.