WORKSHEET SOLUTIONS: CHAIN RULE

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

These solutions have not been properly proofread! Remember that there is extra credit for detecting errors.

Chain rule: If g is differentiable at x and f is differentiable at $g(x)$, and if

$$h(x) = f(g(x))$$

for all x (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Example: If $h(x) = \sin(x^3)$ then write $h(x) = f(g(x))$ with

$$f(u) = \sin(u) \qquad \text{and} \qquad g(x) = x^3.$$

Thus

$$h'(x) = f'(g(x)) \cdot g'(x) = \sin'(x^3) \cdot \frac{d}{dx}\big(x^3\big) = \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

Caution: "$\sin'(x^3)$" means you take the derivative of the sine function and evaluate it at x^3. It does *not* mean $\cancel{\frac{d}{dx}\big(\sin(x^3)\big)}$. The expression $\frac{d}{dx}\big(\sin(x^3)\big)$ means $h'(x)$, the derivative with respect to x of the function whose value at x is $\sin(x^3)$.

Now differentiate the following functions.

1. Let $w(x) = \cos(\sin(x)))$. First, if we want to usefully write $w(x) = f(g(x))$, then

$$f(u) = \cos(u) \qquad \text{and} \qquad g(x) = \sin(x),$$

so

$$f'(u) = -\sin(u) \qquad \text{and} \qquad g'(x) = \cos(x).$$

If $w(x) = \cos(\sin(x)))$ then

$$w'(x) = \cos'(\sin(x)) \cdot \sin'(x) = -\sin(\sin(x))\cos(x).$$

2. Let $p(x) = (2x^3 + 7x + 8)^{28}$. First, if we want to usefully write $p(x) = f(g(x))$, then

$$f(u) = u^{28} \qquad \text{and} \qquad g(x) = 2x^3 + 7x + 8,$$

so

$$f'(u) = 28u^{27} \qquad \text{and} \qquad g'(x) = 6x^2 + 7.$$

Thus

$$\frac{d}{dx}\big((2x^3 + 7x + 8)^{28}\big) = 28(2x^3 + 7x + 8)^{27} \cdot \frac{d}{dx}\big(2x^3 + 7x + 8\big) = 28(2x^3 + 7x + 8)^{27}(6x^2 + 7).$$

(It is wrong if you leave out any of the parentheses.)

3. Let $q(x) = \sqrt{2x^3 + 7x + 8}$. First, if we want to usefully write $q(x) = f(g(x))$, then

$$f(u) = \sqrt{u} = u^{1/2} \qquad \text{and} \qquad g(x) = 2x^3 + 7x + 8,$$

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so

$$f'(u) = \frac{1}{2}\, u^{-1/2} \qquad \text{and} \qquad g'(x) = 6x^2 + 7.$$

Now, if $q(x) = \sqrt{2x^3 + 7x + 8}$ then

$$q'(x) = \frac{1}{2}\,(2x^3 + 7x + 8)^{-1/2} \cdot \frac{d}{dx}\big(2x^3 + 7x + 8\big) = \frac{1}{2}(2x^3 + 7x + 8)^{-1/2}(6x^2 + 7).$$

(It is wrong if you leave out any of the parentheses.) It isn't necessary, but you can rewrite the answer as

$$q'(x) = \frac{6x^2 + 7}{2\sqrt{2x^3 + 7x + 8}}.$$

4. Let $m(x) = \cos(x\sin(x))$. First, if we want to usefully write $m(x) = f(g(x))$, then

$$f(u) = \cos(u) \qquad \text{and} \qquad g(x) = x\sin(x),$$

so, using the **product rule** to find $g'(x)$,

$$f'(u) = -\sin(u) \qquad \text{and} \qquad g'(x) = \sin(x) + x\cos(x).$$

Now, if $m(x) = \cos(x\sin(x))$ then

$$m'(x) = \cos'(x\sin(x)) \cdot \frac{d}{dx}\big(x\sin(x)\big) = -\sin(x\sin(x))\big(\sin(x) + x\cos(x)\big).$$

(It is wrong if you leave out any of the parentheses, except that writing $\sin x$ and $\cos x$ are allowed.)