MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 3

This homework sheet is due **in class** on Wednesday 16 April 2025 (week 3). Write answers on a separate piece of 8.5 by 11 inch paper, well organized and well labelled, with each solution starting on the left margin of the page.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

Point values as indicated; total 50 points.

1. (10 points.) This problem is mostly about using correct notation. Accordingly, most of the credit is for correctness of notation. See the discussion in class Wednesday 9 April.

Consider the problem of finding the derivative of the function $m(t) = 2t^7 - bt^{-2} - \pi^3$, in which b is a *constant*.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. In particular, put "=" and differentiation symbols everywhere they belong, and nowhere else. Start with "$m'(t)$". Use $\frac{d}{dx}$ to indicate differentiation with respect to x, $\frac{d}{dt}$ to indicate differentiation with respect to t, etc., with appropriate parentheses.

Show at least the following steps:

(1) Using the sum, difference, and constant multiple rules for derivatives.
(2) Using the power rule on each of the steps from (1), including the "-1" part.
(3) Simplification of the expression resulting from (2).

Solution. We have

$$m'(t) = 2\frac{d}{dt}\left(t^7\right) - b\frac{d}{dt}\left(t^{-2}\right) + \frac{d}{dt}\left(\pi^3\right) = 2 \cdot 7t^{7-1} - b \cdot (-2)t^{-2-1} + 0 = 14t^6 + 2bt^{-3}.$$

$\square$

Comments. The expression π^3 is a *constant*, so its derivative is zero.

It is wrong to use "$\cancel{\frac{d}{dx}}$", since you are differentiating with respect to t.

The expression "$\cancel{b \cdot -2}$" is wrong: never write two operation symbols next to each other, *especially* subtraction and multiplication. See the information on notation.

No expression like

$$\cancel{14t^6 - bt^{-2} - \pi^3} \qquad \text{or} \qquad \cancel{14t^6 + 2bt^{-3} - \pi^3}$$

may appear anywhere in any correct work: either differentiate all terms or none of them. However,

$$14t^6 - \frac{d}{dt}\left(bt^{-2}\right) + \frac{d}{dt}\left(\pi^3\right)$$

is perfectly legitimate, since for every term, it is the derivative which appears.

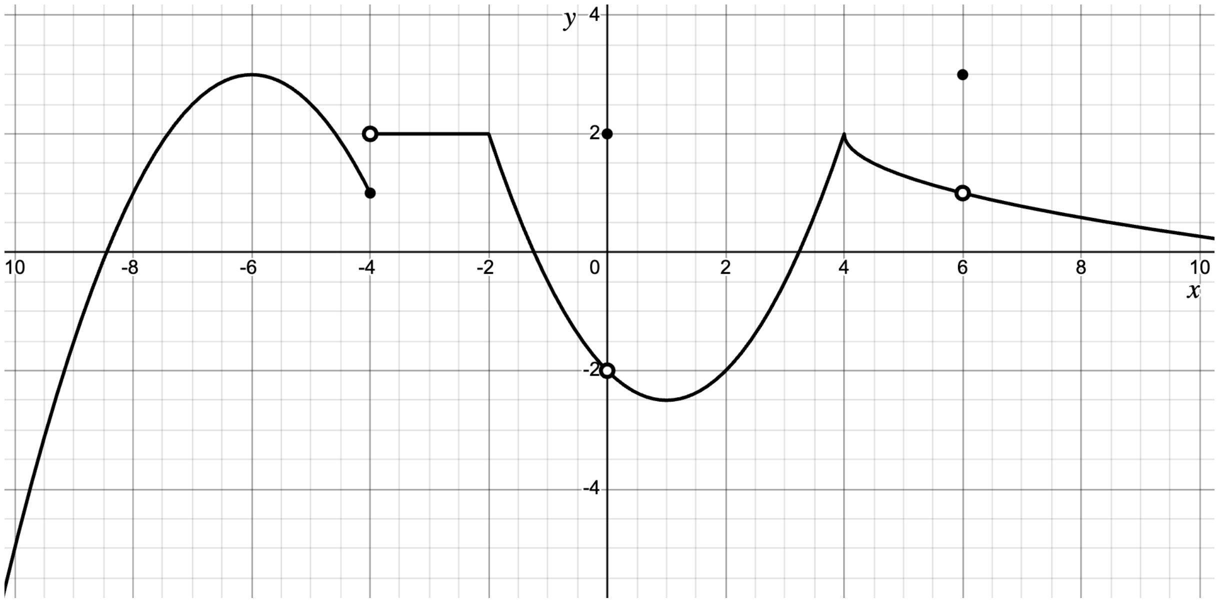
It is acceptable, but not good, to write "$\frac{d}{dt}t^7$". It is wrong to write "$\cancel{\frac{d}{dt}2t^7}$", because this is ambiguous.

For textbook discussion of the next problem, see the graphs on the following pages, and the associated discussion: 135, 140, 143, 145, 151, 180, 182, 235–237, and 239. (We will return to infinite limits.)

It is very important that you understand limits, continuity, and derivatives in terms of the pictures. Otherwise, the pictures I draw on the board during class etc. will not make sense.

Date: 16 April 2025 .

2. (5 points/part) For the function $y = w(x)$ graphed below, answer the following questions.



(a) Does $\lim_{x\to -4} w(x)$ exist? If so, what is it? If not, why not?

Solution: The limit $\lim_{x\to -4} w(x)$ does not exist. Informally, there is a jump in the graph of the function at that point. Formally, $\lim_{x\to -4^+} w(x) = 2$ and $\lim_{x\to -4^-} w(x) = 1$. Since the one sided limits don't agree, the limit does not exist..

It is not true that $\cancel{\lim_{x\to -4} w(x) = 1}$. That is $w(-4)$.

(b) Does $\lim_{x\to 6} w(x)$ exist? If so, what is it? If not, why not?

Solution: You can see from the graph that one can make $w(x)$ as close as one wants to 1 by requiring that x be close enough to 6 but different from 6. Therefore $\lim_{x\to 6} w(x) = 1$.

It is not true that $\cancel{\lim_{x\to 6} w(x) = 3}$. That is $w(6)$.

(c) What is the largest interval containing 2 on which w is continuous? Why?

Solution: $(0, 6)$. It is clear from the graph that w is continuous on this interval: there are no breaks, jumps, or holes. However, w is not continuous at either 0 or 6: $\lim_{x\to 0} w(x) = -2$ but $w(6) = 2$, and $\lim_{x\to 6} w(x) = 1$ but $w(6) = 3$.

(d) Which of the following best describes $w'(-9)$? Why?
(1) $w'(-9)$ does not exist.
(2) $w'(-9)$ is close to 0.
(3) $w'(-9)$ is positive and not close to 0.
(4) $w'(-9)$ is negative and not close to 0.
(5) None of the above.

Solution: $w'(-9)$ is the slope of the tangent line to the graph of $y = w(x)$ at $x = -9$. You can tell from inspection that this slope is positive and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around 6. (In fact, $w'(-9) = 6$. also, it is clear that $w'(x) > 0$ when $-10 < x < -6$.)

(e) Which of the following best describes $w'(-6)$? Why?
(1) $w'(-6)$ does not exist.
(2) $w'(-6)$ is close to 0.
(3) $w'(-6)$ is positive and not close to 0.

(4) $w'(-6)$ is negative and not close to 0.
(5) None of the above.

Solution: $w'(-6)$ is the slope of the tangent line to the graph of $y = w(x)$ at $x = -6$. You can tell from inspection that the tangent line is nearly horizonal, so its slope is close to 0. So choice (2) above is correct. (In fact, for the function used, $w'(-6) = 0$.)

(f) List all points in $(-10, 10)$ at which w is not differentiable. Give reasons.

Solution: The answer is $x = -4$, $x = -2$, $x = 0$, $x = 4$, and $x = 6$. The function w is not differentiable at -4, 0, and 6, since w is not continuous at those points. Also, is not differentiable at -2 and 4, becuase there are corners in the graph at those points, so there is no tangent line.

3. (10 points.) Find the derivative of the function $R(t) = 4at^3 - t^2\cos(t) - \pi^2$. ($a$ is a *constant*.)

Solution: The product rule says

$$\frac{d}{dt}\big(t^2\cos(t)\big) = 2t\cos(t) + t^2(-\sin(t)) = 2t\cos(t) - t^2\sin(t).$$

Also, $\frac{d}{dt}\big(\pi^2\big) = 0$ because π^2 is a constant. Using the power and multiplication by a constant rules on the first term, this gives

$$R'(t) = 12at^3 - [2t\cos(t) - t^2\sin(t)] - 0 = 12at^3 - 2t\cos(t) + t^2\sin(t).$$

4. (10 points.) Find the derivative of the function $h(s) = \sqrt{12 + s^2 + \sin(s)}$.

Solution: We rewrite the function to make it easy to differentiate:

$$h(s) = (12 + s^2 + \sin(s))^{1/2}.$$

Now use the chain rule:

$$h'(s) = \tfrac{1}{2}\big(12 + s^2 + \sin(s)\big)^{1/2-1} \cdot \frac{d}{ds}(12 + s^2 + \sin(s)) = \tfrac{1}{2}\big(12 + s^2 + \sin(s)\big)^{-1/2}(2s + \cos(s)).$$

If you like, you can rewrite this as

$$h'(s) = \frac{2s + \cos(s)}{2\sqrt{12 + s^2 + \sin(s)}},$$

but this is not necessary.