WORKSHEET SOLUTIONS: AN OPTIMIZATION PROBLEM

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

A small decorative box has a square base. It is made with materials costing 6 shekels per square inch for the top, 1 shekel per square inch for the bottom, 2 shekels per square inch for the back, 3 shekels per square inch for the front, 4 shekels per square inch for the right side, and 5 shekels per square inch for the left side. What is the largest volume possible for materials costing a total of 84 shekels?

Solutions have not been properly proofread. Remember that there is extra credit for reporting errors!

Step 1: Draw a picture if possible, and name quantities that are used. (There won't be a picture in the solutions.)

Solution. The missing picture should show a box with rectangular sides. Let the lengths of the edges of the bottom be x and y, and let the height be z. So the front and back edges of both the base and the top should be lebelled x, the right and left edges of both the base and the top should be lebelled y, and the four vertical edges should all be lebelled z,

In addition, let V be the volume of the box, and let C be the total cost of the materials. □

Step 2.1: What is to be minimized or maximized, and which of these is it?

Solution. We want to **maximize** V. □

Step 2.2: What are the restrictions on the values of the variables?

Solution. We must have $x \geq 0$, $y \geq 0$, and $z \geq 0$. □

Step 3: Express the quantity in Step 2.1 in terms of quantities you can vary.

Solution. $V = xyx$. □

Step 4.1: Relate the variables used in Step 3.

Solution. The fact that the base is square says $y = x$.

To get a second relation, use the cost condition. The cost of each side is the area of that side multiplied by the cost per unit area. Therefore the costs, in shekels, of the sides are as follows: top: $6xy$; bottom: xy; back: $2xz$; front: $3xz$; right side: $4yz$; and left side: $5yz$. This gives total cost

$$C = 6xy + xy + 2xz + 3xz + 4yz + yz = 7xy + 5xz + 9yz.$$

Put $y = x$ (from above) and $C = 84$, to get $84 = 7x^2 + 14xz$. □

Step 4.2: Use Step 4.1 to write the quantity in Step 2.1 as a function of a single variable.

Solution. Starting from $84 = 7x^2 + 14xz$, it is much easier to solve for z than to solve for x. Thus:

$$z = \frac{84 - 7x^2}{14x} = \frac{6}{x} - \frac{1}{2}x.$$

Therefore

$$V = xyz = x \cdot x \cdot \left(\frac{6}{x} - \frac{1}{2}x\right) = x\left(6 - \frac{1}{2}x^2\right).$$

So $V(x) = x\left(6 - \frac{1}{2}x^2\right)$. □

Step 5: Find the domain for the function in Step 4.2 that is relevant to the problem.

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Solution. We need $x \geq 0$. Also $y \geq 0$, but, since $x = y$, this tells us nothing new.

Finally $z \geq 0$. This says

$$\frac{6}{x} - \frac{1}{2}x \geq 0.$$

This makes sense only if $x \neq 0$. Since $x \geq 0$, we much have

$$x\left(\frac{6}{x} - \frac{1}{2}x\right) \geq 0.$$

Thus $6 - \frac{1}{2}x^2 \geq 0$, so $12 - x^2 \geq 0$, so $-\sqrt{12} \leq x \leq \sqrt{12}$. Combining this with $x \geq 0$ fives the domain $\left[0, \sqrt{12}\right]$. □

Step 6.1: Find the critical points of the function in Step 4.2.

Solution. For easier differentiation, multiply out: $V(x) = 6x - \frac{1}{2}x^3$. Therefore

$$V'(x) = 6 - \frac{3}{2}x^2 = \frac{3}{2}(4 - x^2).$$

This is zero when $x = 2$ and $x = -2$. □

Step 6.2 (valid if you get a problem on a closed bounded interval): Evaluate the function in Step 4.2 at the numbers from Step 6.1 and the endpoints of the domain found in Step 5.

Solution. Reject the critical point $x = -2$, since it is not in the domain. **(I must see you do this!** Otherwise, I don't know that you correctly solved the equation $V'(x) = 0$.) Then, using the formula $V(x) = x\left(6 - \frac{1}{2}x^2\right)$, (which is easier to use here), evaluate:

$$V(0) = 0, \qquad V\left(\sqrt{12}\right) = 0, \qquad \text{and} \qquad V(2) = 2 \cdot (6 - 2) = 8.$$

(I must see you do this!) □

Step 6.3: Use the outcome of Step 6.2 to find the absolute minimum or maximum, as appropriate, of the function in Step 4.2 on the domain in Step 5. Use this to answer the question asked in the problem.

Solution. This largest of these values is $V(2) = 8$. Therefore the largest possible volume is 8 cubic inches. □