# WORKSHEET SOLUTIONS: IMPLICIT DIFFERENTIATION

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall the chain rule: If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, and if $h(x) = f(g(x))$ for all $x$ (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

1. Consider the problem: use implicit differentiation to find $\dfrac{dy}{dx}$: $x^2 + y^2 = 49$. You **must** solve for $\dfrac{dy}{dx}$. Read the discussion below before trying this problem!

The above is how these problems are written in the textbook and in many WeBWorK problems. It doesn't really tell you what it should, but you need to get used to it. (Sorry: I can't change the textbook or what physicists do.) It means the following.

Assume $y$ is a differentiable function of $x$ given implicitly by $x^2 + [y(x)]^2 = 49$. Find $y'(x)$ in terms of $x$ and $y(x)$.

In particular, in implicit differentiation problems, $y$ (or some other variable) is implicitly a function of $x$ (or some other variable). So, for example, $\dfrac{d}{dx}\big(y^3\big) = 3y^2\dfrac{dy}{dx}$, not zero (and certainly not $\cancel{3y^2}$—that is **never** right).

Also, $\dfrac{dy}{dx}(x^2y + y^6)$ means the product of $\dfrac{dy}{dx}$ and $x^2y + y^6$. It does **not** mean the derivative of $x^2y+y^6$ with respect to $x$ (or with respect to anything else). That is correctly written $\dfrac{d}{dx}\big(x^2y+y^6\big)$. Getting this wrong is serious error, and causes final answers to be badly wrong.

1a. When you differentiate $x^2+y^2 = 49$ or $x^2+[y(x)]^2 = 49$ with respect to $x$, there is one term on which you will need the chain rule. Which term is it?

*Solution.* The term $y^2$ or $[y(x)]^2$, depending on which notation you use. □

1b. Differentiate $x^2 + y^2 = 49$ or $x^2 + [y(x)]^2 = 49$ with respect to $x$, remembering (if you use the first notation) that $y$ is a function of $x$. Show an intermediate step, using $\dfrac{d}{dx}\big(\ldots\big)$ notation.

*Solution.* Version 1: write $y$ explicitly as a function of $x$, getting:

$$x^2 + y(x)^2 = 49$$

Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(y(x)^2\big) = \frac{d}{dx}\big(49\big).$$

$$2x + 2y(x)y'(x) = 0$$

Version 2, in physicists' notation: Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(y2\big) = \frac{d}{dx}\big(49\big).$$

$$2x + 2y\frac{dy}{dx} = 0$$

□

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If your solution contains $\cancel{\frac{dy}{dx}(x^2)}$ or $\cancel{\frac{dy}{dx}(y2)}$, it is wrong!

1c. In the result you got above, solve for $\frac{dy}{dx}$ or $y'(x)$, depending on which notation you use.

*Solution.* Version 1, with $y$ explicitly as a function of $x$:

$$2x + 2y(x)y'(x) = 0$$
$$2y(x)y'(x) = -2x$$
$$y'(x) = \frac{-2x}{2y(x)} = -\frac{x}{y(x)}.$$

Version 2, in physicists' notation:

$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

□

2. If $y^3 = x^2 + \sin(y) + 5$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Steps:

2a. When you differentiate $y^3 = x^2 + \sin(y) + 5$ or $y^3 = x^2 + \sin(y(x)) + 5$ with respect to $x$, there are two terms on which you will need the chain rule. Which terms are they?

*Solution.* The terms $y^3$ and $\sin(y)$, or $[y(x)]^3$ and $\sin(y)$, depending on which notation you use. □

2b. Differentiate $y^3 = x^2 + \sin(y) + 5$ or $y^3 = x^2 + \sin(y(x)) + 5$ with respect to $x$, remembering (if you use the first notation) that $y$ is a function of $x$. Show an intermediate step, using $\frac{d}{dx}(\ldots)$ notation.

*Solution.* Version 1, with $y$ explicitly as a function of $x$:

$$y(x)^3 = x^2 + \sin(y(x)) + 5$$

Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(y(x)^3\big) = \frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(\sin(y(x))\big) = \frac{d}{dx}(5).$$
$$3y(x)^2y'(x) = 2x + \cos(y(x))y'(x)$$

Version 2, in physicists' notation: Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(y^3\big) = \frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(\sin(y)\big) = \frac{d}{dx}(5).$$
$$3y^2\frac{dy}{dx} = 2x + \cos(y)\frac{dy}{dx}$$

□

If your solution contains $\cancel{\frac{dy}{dx}(y3)}$, $\cancel{\frac{dy}{dx}(x^2)}$, or $\cancel{\frac{dy}{dx}(\sin(y))}$, it is wrong!

2c. In the result you got above, solve for $\frac{dy}{dx}$ or $y'(x)$, depending on which notation you use.

*Solution.* Version 1, with $y$ explicitly as a function of $x$:

$$\begin{aligned} 3y(x)^2y'(x) &= 2x + \cos(y(x))y'(x) \\ 3y(x)^2y'(x) - \cos(y(x))y'(x) &= 2x \\ \big(3y(x)^2 - \cos(y(x))\big)y'(x) &= 2x \\ y'(x) &= \frac{2x}{3y(x)^2 - \cos(y(x))} \end{aligned}$$

Version 2, in physicists' notation:

$$\begin{aligned} 3y^2\frac{dy}{dx} &= 2x + \cos(y)\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} - \cos(y)\frac{dy}{dx} &= 2x \\ (3y^2 - \cos(y))\frac{dy}{dx} &= 2x \\ \frac{dy}{dx} &= \frac{2x}{3y^2 - \cos(y)}. \end{aligned}$$

□