MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 7 PART 2.

This homework sheet is due in class on Friday 16 May 2025 (week 7), in class. Write answers on a separate piece of paper, well organized and well labelled, with **each solution starting on the left margin of the page**.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

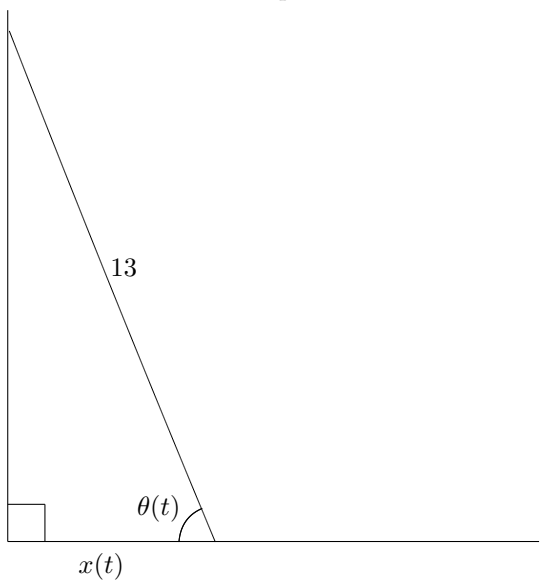
20 points.

A 13 foot ladder leans against a vertical wall in a room with a high ceiling and level floor. Because the floor is slippery, the foot of the ladder is sliding away from the wall. When the foot is 5 feet from the wall, it is sliding away at 3 feet per hour. At this time, what is the rate of change of the angle between the ladder and the floor? (Be sure to include the correct units.)

Comments. This problem involves an angle, and in this way is different from examples done so far on worksheets or in lecture. But the basic principles are the same.

Example 4.3 in the textbook also involves an angle, with similar geometry. The algebra at the end of this problem is simpler than in Example 4.3.

Solution. Here is a picture, with the quantities we need to use labelled, and the proportions approximately correct for the time in the problem:



As shown, let $x(t)$ be the distance of the foot of the ladder from the base of the wall, in feet. Let $\theta(t)$ be the angle between the ladder and the floor, in radians. (Remember that we *always* use radians.) Recall that the ladder is 13 feet long, and this length does not change. (We could also label the vertical side; as it turns out, we don't need to.)

Let t_0 be the time at which we are interested. We are given the following:

$$x(t_0) = 5 \qquad \text{and} \qquad x'(t_0) = 3.$$

(The derivative $x'(t_0)$ is positive because the distance from the foot of the ladder to the wall is increasing.)

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Our quantities are related by the equation $x(t) = 13\cos(\theta(t))$, since we have a right triangle. Differentiating this equation with respect to t gives:

$$x'(t) = -13\sin(\theta(t))\theta'(t).$$

(Don't forget to use the chain rule!) Substituting t_0 for t gives:

$$x'(t_0) = -13\sin(\theta(t_0))\theta'(t_0).$$

Substituting for the known quantities gives:

$$3 = -13\sin(\theta(t_0))\theta'(t_0).$$

We want to find $\theta'(t_0)$, so we still need $h(t_0)$. Substituting for the known quantities in the equation $x(t_0) = 13\cos(\theta(t_0))$ gives $5 = 13\cos(\theta(t_0))$, so $\cos(\theta(t_0)) = \frac{5}{13}$. Then

$$\sin(\theta(t_0)) = \sqrt{1 - \left(\tfrac{5}{13}\right)^2} = \tfrac{12}{13}.$$

(We use the positive square root, because $0 < \theta(t) < \frac{\pi}{2}$.) Therefore

$$3 = -13 \cdot \tfrac{12}{13}\theta'(t_0) = -12\theta'(t_0),$$

whence

$$\theta'(t_0) = -\tfrac{1}{4}.$$

So the angle is decreasing at $\frac{1}{4}$ radians per hour. (The units are required.)

Here, for reference, is what the solution looks like in physicists' notation.

We have $x = 13\cos(\theta)$. Differentiate with respect to t:

$$\frac{dx}{dt} = -13\sin(\theta)\frac{d\theta}{dt}.$$

(Don't forget the factor $\dfrac{d\theta}{dt}$!) Use the equation $x = 13\cos(\theta)$ and the fact that $x = 5$ at the time of interest to find that $\cos(\theta) = \frac{5}{13}$ at the time of interest, and use this to show, as above, that $\sin(\theta) = \frac{12}{13}$ at the time of interest. Then substitute $x = 5$, $\sin(\theta) = \frac{12}{13}$, and $\dfrac{dx}{dt} = 5$, getting

$$3 = -13 \cdot \left(\frac{12}{13}\right)\frac{d\theta}{dt},$$

and solve for $\dfrac{d\theta}{dt}$ the same way we solved for $\theta'(t_0)$ above. $\square$

The relation can also be written as

$$\theta(t) = \arccos\left(\frac{x(t)}{13}\right).$$

Doing things this way, you will need to use the chain rule and

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

This is messier, but does avoid finding $\sin(\theta(t_0))$.