WORKSHEET SOLUTIONS: INFLECTION POINTS; TANGENT LINES 1

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall:

- If $f''(x) > 0$ on an interval (a, b), then f is concave up on (a, b).
- If $f''(x) < 0$ on an interval (a, b), then f is concave down on (a, b).

1. Let $f(x) = x^4$, so $f''(x) = 12x^2$, and $f''(x) = 0$ only when $x = 0$. In the following steps, you will decide whether f has an inflection point at $x = 0$.

(a) What is the sign of $f''(x)$ on $(0, \infty)$?

Solution. $f''(x) = x^2 > 0$ when $x > 0$. □

(b) Is f concave up or concave down on $(0, \infty)$?

Solution. f is concave up on $(0, \infty)$ because $f''(x) > 0$ on $((0, \infty)$. □

(c) What is the sign of $f''(x)$ on $(-\infty, 0)$?

Solution. $f''(x) = x^2 > 0$ when $x > 0$. □

(d) Is f concave up or concave down on $(-\infty, 0)$?

Solution. f is concave up on $(-\infty, 0)$ because $f''(x) > 0$ on $((-\infty, 0)$. □

(e) Does f have an inflection point at $x = 0$?

Solution. No, because the concavity does not change. □

2. Let $f(x) = x^4 - 6x^3$. Find the intervals of concavity and the inflection points. Start by finding $f''(x)$. Check where $f''(x)$ is zero, where $f''(x)$ is strictly positive, and where $f''(x)$ is strictly negative.

Solution. First,

$$f'(x) = 4x^3 - 18x^2 \qquad \text{and} \qquad f''(x) = 12x^2 - 36x = 12x(x - 3).$$

So $f''(x) = 0$ when $x = 0$ and $x = 3$. Therefore there are possible inflection points at $x = 0$ and $x = 3$.

On the interval $(-\infty, 0)$, we have $f''(x) > 0$. (Since f'' is continuous, we can check at one point, say $x = -1$, which gives $f''(-1) = 12(-1)(-4) > 0$.)

Similarly, on the interval $(0, 3)$, we have $f''(x) < 0$, and on the interval $(3, \infty)$, we have $f''(x) > 0$. Therefore:

- f is concave up on $(-\infty, 0)$.
- f is concave down on $(0, 3)$.
- f is concave up on $(3, \infty)$.
- f has an inflection point at $x = 0$, because concavity changes from up to down.
- f has an inflection point at $x = 3$, because concavity changes from down to up.

□

3. A fire-breathing monster was asked to find the equation of the tangent line to the graph of $g(x) = x^2 + 2x$ at $x = 2$. Its answer was $y = (2x + 2)(x - 2) + 8$.

Even without knowing the formula for g, this answer is *obviously* wrong. Why is it obviously wrong?

Solution. $y = (2x + 2)(x - 2) + 8$ is not the equation of **any** line, since it is quadratic.

You can see this more explicitly by expanding:

$$(2x + 2)(x - 2) + 8 = 2x^2 - 2x + 4.$$

The equation $y = 2x^2 - 2x + 4$ is certainly not the equation of **any** line, □

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