SOLUTIONS TO MIDTERM 1 RETAKE PROBLEMS FOR 11:00 AM

1. (1 point.) True or false: Inverse functions were invented by the Devil.

Solution: Well, some people think so. But wait until you see tensor products!

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2. (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

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3. (a) (9 points) If $f(x) = 5x - x^2$, compute the derivative $f'(4)$ *directly from the definition of the derivative* (which you are supposed to have given above). (No credit will be given for just using the differentiation rules, but see Part (b.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} f'(4) &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{5(4+h) - (4+h)^2 - (5 \cdot 4 - 4^2)}{h} \\ &= \lim_{h \to 0} \frac{20 + 5h - (16 + 8h + h^2) - (20 - 16)}{h} \\ &= \lim_{h \to 0} \frac{20 + 5h - 16 - 8h - h^2 - 20 + 16}{h} = \lim_{h \to 0} \frac{-3h - h^2}{h} = \lim_{h \to 0} (-3 - h) = -3. \end{aligned}$$

(b) (1 point) Use the differentiation rules we have learned to check your answer to part (a).

Solution: $f'(x) = 5 - 2x$, so $f'(4) = 5 - 2 \cdot 4 = -3$.

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4. (10 points) Differentiate te function $L(x) = \dfrac{\sin(x)}{e^x + x^4 + 15} + \ln(2)$. (You need not do this directly from the definition.)

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Solution: The second term is a constant, so its derivative is zero. On the first term, use the quotient rule:

$$\begin{aligned} L'(x) &= \frac{\sin'(x)(e^x + x^4 + 15) - \sin(x)\frac{d}{dx}\bigl(e^x + x^4 + 15\bigr)}{(e^x + x^4 + 15)^2} \\ &= \frac{\cos(x)(e^x + x^4 + 15) - \sin(x)(e^x + 4x^3)}{(e^x + x^4 + 15)^2}. \end{aligned}$$

Parentheses are required in the several places. For example, if

$$\cancel{\frac{d}{dx}e^x + x^4} + 15, \qquad \cancel{\cos(x)e^x + x^4} + 15, \qquad \cancel{-\sin(x)e^x + 4x^3},$$

or anything similar appears in your work, it is wrong.

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5. (10 points) Differentiate the function $h(x) = (x^5 + 3x)^8$. (You need not do this directly from the definition.)

Solution: Use the chain rule and the power rule:

$$\begin{aligned} h'(x) &= 8(x^5 + 3x)^7 \cdot \frac{d}{dx}\bigl(x^5 + 3x\bigr) \\ &= 8(x^5 + 3x)^7 \cdot (5x^4 + 3). \end{aligned}$$

Parentheses are required in the first and second steps: if

$$\cancel{\frac{d}{dx}x^5} + 3x, \qquad \cancel{8(x^5 + 3x)^7 \cdot 5x^4} + 3,$$

or anything similar appears in your work, it is wrong. Also, remember that $\frac{d}{dx}\bigl((x^5 + 3x)^8\bigr)$ means the derivative of the entire function, not just the derivative of the 8th power part, so the expressions

$$\cancel{\frac{d}{dx}\bigl((x^5 + 3x)^8\bigr)\frac{d}{dx}\bigl(x^5 + 3x\bigr)} \qquad \text{or} \qquad \cancel{\frac{d}{dx}\bigl((x^5 + 3x)^8\bigr)\bigl(5x^4 + 3\bigr)}$$

are not part of any correct solution.

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6. (10 points) Find the exact value of $\displaystyle\lim_{x \to -2} \frac{x^2 - 4x - 12}{7(x + 2)}$, or explain why this limit does not exist.

Solution: This limit has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the numerator, cancel common factors, and then use the limit laws:

$$\begin{aligned} \lim_{x \to -2} \frac{x^2 - 4x - 12}{7(x + 2)} &= \lim_{x \to -2} \frac{(x - 6)(x + 2)}{7(x + 2)} = \lim_{x \to -2} \frac{x - 6}{7} \\ &= \frac{(\lim_{x \to -2} x) - 6}{7} = \frac{-2 - 6}{7} = -\frac{8}{7}. \end{aligned}$$

Any solution containing

$$\cancel{= \frac{0}{0}}, \qquad \cancel{\lim_{x \to -2} =}, \qquad \text{or} \qquad \cancel{\lim_{x \to -2} \frac{8}{7}}$$

uses incorrect notation, and will therefore lose points.

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7. (20 points.) A rectangular box is to have a base that is twice as long as it is wide. 12 square feet of material are available to make the box (including the top). Find the dimensions of the box which maximizes the volume.

Include units. Be sure to verify that your maximum or minimum really is what you claim it is, using methods we have see so far (not the second derivative test).

Solution: Note: No picture is included in this file. A picture may be provided separately.

We want to maximize the volume of the box; call it V. Let the dimensions of the box be l feet long, w feet wide, and h feet high. Then $V = lwh$.

One relation comes from the assumption that the base is twice as long as it is wide. This says $l = 2w$. Also, the surface area is 12 square feet. (We may assume that the surface area is not less than 12 square feet, since clearly there is no benefit from not using all the material.) The front and back have area lh each, the two ends have area wh each, and the top and bottom have area lw each. The total surface area is $2lh + 2wh + 2lw$. Thus $2lh + 2wh + 2lw = 12$.

The easiest thing to do first is to substitute $2w$ for l everywhere. This gives

$$V = 2w^2h \qquad \text{and} \qquad 12 = 4wh + 2wh + 4w^2 = 6wh + 4w^2.$$

Next, we solve the second equation for one of the variables and substitute in the first equation. Clearly it is easier to solve for h:

$$h = \frac{12 - 4w^2}{6w} = \frac{6 - 2w^2}{3w}.$$

Substitute, and use function notation since we have one variable and want to differentiate:

$$V(w) = 2w^2h = 2w^2 \cdot \frac{6 - 2w^2}{3w} = 4w - \tfrac{4}{3}w^3.$$

(Simplify before differentiating!)

Clearly $w \geq 0$. Also $h \geq 0$. Since $h = \frac{6-2w^2}{3w}$ and $3w \geq 0$, this means $6 - 2w^2 \geq 0$. Thus $w \leq \sqrt{3}$. (Also, this tells us that $w \geq -\sqrt{3}$, but we already know $w \geq 0$.) We therefore must maximize $V(w) = 4w - \frac{4}{3}w^3$ subject to the condition $0 \leq w \leq \sqrt{3}$.

We look for critical numbers: $V'(w) = 4 - 4w^2$, and this is zero when $w^2 = 1$, that is, $w = \pm 1$. We reject $w = -1$ because -1 is not in $[0, \sqrt{3}]$. (I must see you do this!) But $w = 1$ is in $[0, \sqrt{3}]$.

We are working over a closed bounded interval, so we can compare values at the critical numbers and the endpoints. We check:

$$V(0) = 0, \qquad V(1) = 4 \cdot 1 - \tfrac{4}{3} \cdot 1^3 = \tfrac{8}{3}, \qquad \text{and} \qquad V(\sqrt{3}) = 0.$$

Obviously $V(1)$ is the largest. Thus

$$w = 1, \qquad l = 2w = 2, \qquad \text{and} \qquad h = \frac{6 - 4w^2}{3w} = \tfrac{4}{3}.$$

So the dimensions are 2 feet by 1 foot by $\frac{4}{3}$ feet. (Be sure to include the units!)

8. (13 points) Use the methods of calculus to find the exact values of x at which the function $q(x) = x^3 - 12x + 20$ takes its absolute minimum and maximum on the interval $[-3, 0]$.

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate q at all critical numbers in the interval and at the endpoints, and compare values.

To find the critical numbers, we differentiate q, solve the equation $q'(x) = 0$, and find all numbers x in our interval such that $q'(x)$ does not exist. The derivative of q is

$$q'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

It exists everywhere, and it is zero when $x = -2$ and $x = 2$.

Since $x = 2$ is not in $[-3, 0]$, we ignore it. (**I must see you reject 2.** If I don't see this, I will assume you didn't correctly solve the equation $q'(x) = 0$.)

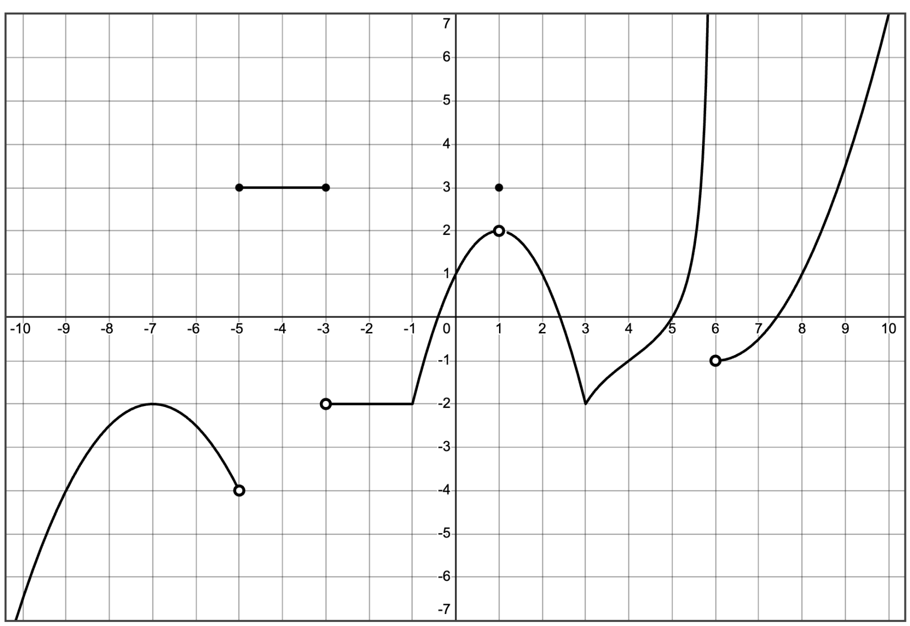
We now have one critical number, namely -2. So we must compare the values of q at -2, and at the endpoints -3 and 0.

Since $q(-2) = 36$, $q(-3) = 29$, and $q(0) = 20$, the smallest of these is $q(0)$ and the largest is $q(-2)$, So the absolute minimum on the interval $[-3, 0]$ occurs at $x = 0$ and the absolute maximum on the interval $[-3, 0]$ occurs at $x = -2$.

Note that $x = 2$ is not correct for the minimum, even though $q(2) = 4$, because 2 is not in the interval $[-3, 0]$.

No credit will be given for any solution which does not show evidence of an attempt to find the critical numbers of q. In particular, no credit will be given for simply comparing the values of q at the integers in the interval.

9. (5 points/part) For the function $y = L(x)$ graphed below, answer the following questions:



(a) Does $\lim_{x \to 1} L(x)$ exist? If so, what is it? If not, why not?

Solution: $\lim_{x\to 1} L(x) = 2$. You can see from the graph that when x is close enough to 1 (but $x \neq 1$, then $L(x)$ is close to 2.

The answer is not $\not 3$. That is $L(1)$.

(b) Is L continuous at -1? Why or why not?

Solution: The function L is continuous at -1, because $L(-1)$ and $\lim_{x\to -1} L(x)$ are both equal to -2. In particular, they are equal.

Informally, the graph can be drawn near $x = -1$ without lifting the pencil off the paper.

(It is true that L is not differentiable at $x = -1$, because of the corner, but that isn't what was asked.)

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10. (10 points.) This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\displaystyle\lim_{x\to -1} \frac{x^3 - 2x^2 - 3x}{x+1}$. The method is to factor the numerator and cancel one of the factors. The factors of the numerator are $x+1$, x, and $x-3$. (You need not check these.)

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. In particular, put "=" and "lim" everywhere they belong, and nowhere else. Start by writing $\displaystyle\lim_{x\to -1} \frac{x^3 - 2x^2 - 3x}{x+1}$. Show at least the following steps (not labelled):

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x = -1$.
(3) After substituting $x = -1$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified.

Solution.

$$\begin{aligned}\lim_{x\to -1} \frac{x^3 - 2x^2 - 3x}{x+1} &= \lim_{x\to -1} \frac{(x+1)x(x-3)}{x+1} \\ &= \lim_{x\to -1} x(x-3) = (-1)(-1-3) = (-1)(-4) = 4.\end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq -1$, we have

$$\frac{x^3 - 2x^2 - 3x}{x+1} = \frac{(x+1)x(x-3)}{x+1} = x(x-3).$$

Therefore

$$\lim_{x\to -1} \frac{x^3 - 2x^2 - 3x}{x+1} = \lim_{x\to -1} x(x-3) = (-1)(-1-3) = (-1)(-4) = 4.$$

This completes the solution. □

Comments. In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

The symbol "$\lim_{x\to-1}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\cancel{\lim_{x\to-1} x(x-3) = \lim_{x\to-1}(-1)(-1-3)}$$

is a mathematically true statement, but does not correctly show the intended step.

Every parenthesis shown is essential, except that $(-1)(-1-3)$ could be $-1(-1-3)$ and $(-1)(-4)$ could be $-(-4)$. In particular, $\cancel{-1\cdot -4}$ is wrong. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.