SOLUTIONS TO THE SAMPLE MIDTERM 1, MATH 251 (PHILLIPS), WINTER 2025

Contents

1. Midterm 1 Information

At least 80% of the points on the real exam will be modifications of problems from Midterms 1 and 2 from the last time I taught the course, the problems below, homework problems (particularly written homework), and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be differentiated and of the limits to be computed could vary substantially, and the methods required to do them might occur in different combinations. Word problems could have rather different descriptions, but similar methods will be used.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation for limits etc. in the book, in handouts, in files posted on the course website, and on the blackboard; *use it*. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Here is the instruction sheet for Midterm 1:

(1) DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
(2) The exam pages are **two sided**.
(3) Closed book, except for a 3×5 file card, written on both sides.
(4) The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.
(5) The point values are as indicated in each problem; total 100 points.
(6) Write all answers on the test paper. Use the back of the last page for long answers or scratch work. (If you do write an answer there, indicate on the page containing the problem where your answer is.)
(7) Show your work. You must state what you did, legibly, clearly, correctly, and using correct notation. Among many other things, this means putting "=", limit symbols, etc. in all places where they belong, and not in any places where they don't belong. It also means organizing your work so that the order of the steps is clear, and it is clear how the steps are related to each other.

(8) Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, or for which the work is riddled with notation errors, will receive little or no credit.
(9) Say what you mean. Credit will be based on what you say, not what you mean.
(10) When exact values are specified, give answers such as $\frac{1}{7}$, $\sqrt{2}$, $\ln(23)$, or $\frac{2\pi}{9}$. Decimal approximations will not be accepted.
(11) Final answers must always be simplified unless otherwise specified.
(12) Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned (usually by the Tuesday after the exam).
(13) Time: 50 minutes.

2. Sample Midterm 1

1. (a) (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (13 points) If $f(x) = \dfrac{1}{8 - x}$, compute the derivative $f'(2)$ *directly from the definition.* (You can check your answer using a differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient. To handle the expression that appears in the difference quotient, we subtract the fractions in the numerator and then cancel common factors in the numerator and denominator:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{8-(2+h)} - \frac{1}{8-2}}{h} = \lim_{h \to 0} \frac{\frac{1}{6-h} - \frac{1}{6}}{h}$$
$$= \lim_{h \to 0} \frac{\left(\frac{6-(6-h)}{6(6-h)}\right)}{h} = \lim_{h \to 0} \frac{\left(\frac{h}{6(6-h)}\right)}{h} = \lim_{h \to 0} \frac{1}{6(6-h)} = \frac{1}{36}.$$

We can check using the differentiation formulas. It is easiest to use the chain rule: $f(x) = (8 - x)^{-1}$, so $f'(x) = -(8 - x)^{-2} \cdot (-1) = (8 - x)^{-2}$, whence $f'(2) = \frac{1}{36}$. It can also be done using the quotient rule. (However, you get no credit if this is the only thing you do.)

(c) (1 point) Check your answer to Part (b) using a differentiation formula.

Solution: Rewrite $f(x) = (8 - x)^{-1}$, and use the power rule and the chain rule to get $f'(x) = -(8 - x)^{-2}(-1) = (8 - x)^{-2}$. Therefore $f'(2) = (8 - 2)^{-2} = \frac{1}{36}$.

2. (10 points) Let g be a function such that $g'(x) = \sqrt[3]{3x^2-6} - 13$. Differentiate the function $f(t) = \sqrt{e} - t^2 g(t)$. (Your answer might involve the function g. You need not do this directly from the definition.)

Solution: Use the product rule:

$$f'(t) = -\big(2tg(t) + t^2 g'(t)\big) = -2tg(t) - t^2\big(\sqrt[3]{3t^2-6} - 13\big) = -2tg(t) - t^2\sqrt[3]{3t^2-6} + 13t^2.$$

Parentheses are required in the fist step: if $\cancel{-2tg(t) + t^2 g'(t)}$ or similar appears in your work, it is wrong.

The last step is not necessary.

Note that $\sqrt{e}$ is a *constant*, so its derivative is zero.

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3. (10 points) Differentiate the function $h(x) = \sin(6x^2 - 11x)$. (You need not do this directly from the definition.)

Solution: Use the chain rule:

$$h'(x) = \cos(6x^2 - 11x) \cdot \frac{d}{dx}\big(6x^2 - 11x\big) = \cos(6x^2 - 11x) \cdot (6 \cdot 2x - 11) = (12x - 11)\cos(6x^2 - 11x).$$

Parentheses are required in the first and second steps: if

$$\cancel{\frac{d}{dx} 6x^2 - 11x}, \qquad \cancel{\cos(6x^2 - 11x) \cdot 6 \cdot 2x - 11},$$

or anything similar appears in your work, it is wrong.

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4. (20 points) A right circular cylinder is inscribed in a hemisphere (*not* a sphere) of radius $2\sqrt{3}$ inches. Find the largest possible volume of such a cylinder.

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: There is no picture in this file.

We are supposed to maximize the volume of the cylinder. Let's call it V.

For the purpose of description, let's decide that the cylinder is supposed to be inscribed vertically, that is, with the curved surface vertical and with the flat (circular) ends at the top and bottom. Further let x be the distance from the center of the sphere to the top of the cylinder. Note that this is *half* the height of the cylinder. (Other choices are of course possible here; the final answer will be the same.) Let y be the radius of the cylinder, which is the distance from the center of the sphere to the side of the cylinder. Then V is the height $2x$ multiplied by the area πy^2 of the base, that is, $V = 2x \cdot \pi y^2 = 2\pi x y^2$.

There are too many variables in this formula; we must relate x and y. The Pythagorean Theorem gives $x^2 + y^2 = r^2$. Since y^2 already appears in the formula above, but y by itself does not, the easiest way to proceed is:

$$y^2 = r^2 - x^2$$
$$V = 2\pi x y^2 = 2\pi x(r^2 - x^2)$$

Rewriting it for easy differentiation:

$$V(x) = 2\pi r^2 x - 2\pi x^3.$$

(See below for other possibilities.)

Now we consider the allowed values of x. Clearly $x \geq 0$. Also $x \leq r$, because otherwise the cylinder would stick outside the sphere. Any value of x in the interval $[0, r]$ gives a reasonable value of y, and therefore an inscribed cylinder, so these are the only restrictions. (Note: We allow the "degenerate" cases $x = 0$, in which case our "cylinder" is a vertical line, and $x = r$, in which case our "cylinder" is a horizontal disk, so as to be able to use the shortcut for maximization on closed bounded intervals. See below.)

Our problem is now to find the maximum value of $V(x) = 2\pi r^2 x - 2\pi x^3$ for x in the interval $[0, r]$. We search for critical numbers. We have $V'(x) = 2\pi r^2 - 6\pi x^2$ (remember that r is a constant!), so we solve:

$$\begin{aligned} 2\pi r^2 - 6\pi x^2 &= 0 \\ r^2 - 3x^2 &= 0 \\ x^2 &= \tfrac{1}{3} r^2 \\ x &= \pm\sqrt{\tfrac{1}{3} r^2} = \pm\sqrt{\tfrac{1}{3}} \cdot r. \end{aligned}$$

We reject the value $x = -\sqrt{\frac{1}{3}} \cdot r$, because it is not in the interval $[0, r]$. This leaves one critical number, namely $x = \sqrt{\frac{1}{3}} \cdot r$.

Since we are maximizing a continuous function on a closed bounded interval, we can simply compare function values at the critical numbers and endpoints. (This is why we included the degenerate cases $x = 0$ and $x = r$ above.) We compare (using the formula $V(x) = 2\pi x(r^2 - x^2)$, because it is easier to evaluate):

$$V(0) = 0, \qquad V\left(\sqrt{\tfrac{1}{3}} \cdot r\right) = 2\pi\sqrt{\tfrac{1}{3}} \cdot r\left(r^2 - \tfrac{1}{3}r^2\right) = 2\pi\sqrt{\tfrac{1}{3}}\left(1 - \tfrac{1}{3}\right)r^3 = 2\pi\tfrac{2}{3}\sqrt{\tfrac{1}{3}} \cdot r^3,$$

and

$$V(r) = 2\pi r(r^2 - r^2) = 0.$$

Clearly $V\left(\sqrt{\frac{1}{3}} \cdot r\right)$ is the largest. Thus, the maximum volume is $V\left(\sqrt{\frac{1}{3}} \cdot r\right) = 2\pi\frac{2}{3}\sqrt{\frac{1}{3}} \cdot r^3$. (Units are not given, so we can't put them in. You can tell from this that the problem was invented by a mathematician, not a physicist.) This completes the (most direct) solution of the problem.

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5. (10 points) Find the exact value of the limit $\displaystyle\lim_{x \to 1} \frac{x - 1}{x^2 - x - 6}$. Give reasons.

Solution: Both the numerator and denominator are continuous at 1, and the denominator is not zero there. Therefore the limit can be evaluated by simply substituting $x = 1$. That is,

$$\lim_{x \to 1} \frac{x - 1}{x^2 - x - 6} = \frac{1 - 1}{1^2 - 1 - 6} = 0.$$

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6. (10 points) Find the exact value of the limit $\displaystyle\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$. Give reasons.

Solution: This limit has the indeterminate form "$\frac{0}{0}$", so work is needed. Factor the top, and cancel the common factor:

$$\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = \lim_{x\to 3}(x-2) = 1.$$

Any solution containing

$$\cancel{= \frac{0}{0}}, \qquad \cancel{\lim_{x\to 3} =}, \qquad \cancel{\lim_{x\to 3} x-2}, \qquad \text{or} \qquad \cancel{\lim_{x\to 3} 1}$$

uses incorrect notation, and will therefore lose points.

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7. (12 points) Use the methods of calculus to find the exact values of x at which the function $k(x) = 8x^3 - 36x^2 + 30x$ takes its absolute minimum and maximum on the interval $[0, 1]$.

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we differentiate k and solve the equation $k'(x) = 0$. The derivative of k is

$$k'(x) = 24x^2 - 72x + 30.$$

To find the roots, either use the quadratic formula or factor it as

$$24x^2 - 72x + 30 = 6(2x-1)(2x-5).$$

Either way, the roots are $x = \frac{1}{2}$ and $x = \frac{5}{2}$.

We now have two critical numbers, namely $\frac{1}{2}$ and $\frac{5}{2}$. Of these, $\frac{5}{2}$ is not in the interval under consideration, so we ignore it. (**I must see you reject $\frac{5}{2}$.** If I don't see this, I will assume you didn't correctly solve the equation $k'(x) = 0$.) So we must compare the values of k at $\frac{1}{2}$, and at the endpoints 0 and 1. We evaluate:

$$k(0) = 0, \qquad k(1) = 2, \qquad \text{and} \qquad k\left(\tfrac{1}{2}\right) = 7.$$

The largest of these is $k\left(\frac{1}{2}\right)$ and the smallest is $k(0)$, so the absolute maximum of k on the interval $[0, 1]$ occurs at $x = \frac{1}{2}$ and the absolute minimum of k on the interval $[0, 1]$ occurs at $x = 0$. Note that $x = \frac{5}{2}$ is not correct for the minimum, even though $k\left(\frac{5}{2}\right) = -25$, because $\frac{5}{2}$ is not in the interval $[0, 1]$.

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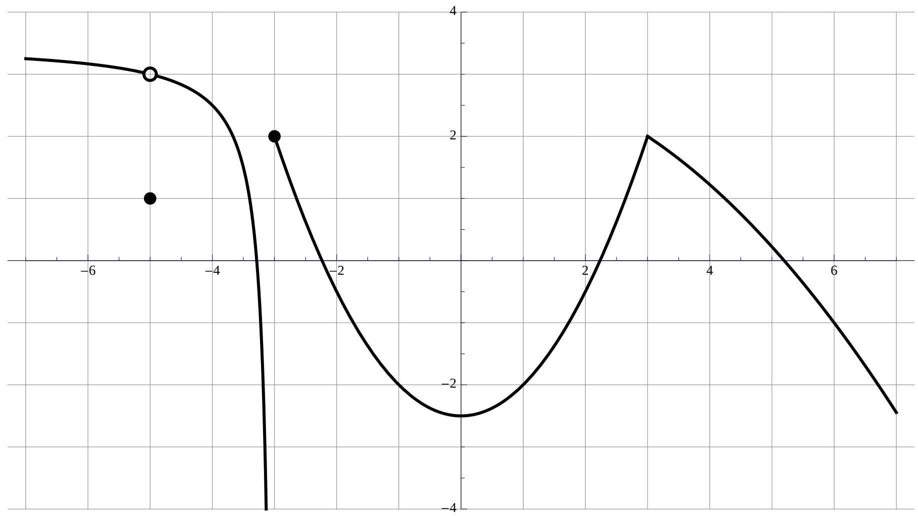
8. For the function $y = k(x)$ graphed below, answer the following questions:



(a) (4 points.) Find $\lim_{x \to -5} k(x)$.

Solution: $\lim_{x \to -5} k(x) = 3$. (Note that it is different from $f(-5) = 1$.)

(b) (4 points.) Which of the following best describes $k'(4)$?
(1) $k'(4)$ does not exist.
(2) $k'(4)$ is close to 0.
(3) $k'(4)$ is positive and not close to 0.
(4) $k'(4)$ is negative and not close to 0.

Solution: $k'(4)$ is the slope of the tangent line to the graph of $y = k(x)$ at $x = 4$. You can tell from inspection that this slope is negative and not close to 0 (choice (4) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around $-8/9$. (In fact, it is clear that $k'(x) < 0$ for $3 < x < 7$.)

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3. Extra Sample Problems for Midterm 1

9. (4 points/part) The following questions refer to the function whose graph is shown in problem 8.

(a) List all numbers a in $(-7, 7)$ such that h is not differentiable at a. Give reasons.

Solution: The answer is $a = -5$, $a = -3$, and $a = 3$. The function h is not differentiable at -5 and at -3, because h is not continuous at these places. It is not differentiable at 3 because there is a corner in the graph, so that there is no tangent line there.

(b) List all numbers a in $(-7, 7)$ such that k is continuous at a but not differentiable at a. Give brief reasons.

Solution: The answer is $a = 3$. The function k is not differentiable at 3 because there is a corner in the graph, so that there is no tangent line there. But clearly k is continuous at 3.

(c) List all numbers a in $(-7, 7)$ such that k is differentiable at a but not continuous at a. Give brief reasons.

Solution: There are no such numbers, because, for any function f and any a, if f is differentiable at a then f is continuous at a.

(d) Which of the following best describes $k'(7)$?
(1) $k'(7)$ does not exist.
(2) $k'(7)$ is close to 0.
(3) $k'(7)$ is positive and not close to 0.
(4) $k'(7)$ is negative and not close to 0.

Solution: $k'(7)$ is the slope of the tangent line to the graph of $y = k(x)$ at $x = 7$. You can tell from inspection that this slope is negative and not close to 0 (choice (4) above). (This is slightly awkward because 7 is on the edge of the part of the graph which is shown.)

(e) Find the largest interval containing 5 on which k is continuous.

Solution: The largest interval containing 5 on which h is continuous is $(-3, 7]$. (Note that k is not continuous at -3, because $\lim_{x\to -3} k(x)$ does not exist.)

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10. (10 points) Find the exact value of the limit $\displaystyle\lim_{x\to 10} \frac{x-10}{3\left(\sqrt{x}-\sqrt{10}\right)}$. Give reasons.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We rationalize the numerator, and then cancel common factors:

$$\begin{aligned}\lim_{x\to 10} \frac{x-10}{3\left(\sqrt{x}-\sqrt{10}\right)} &= \lim_{x\to 10} \frac{(x-10)\left(\sqrt{x}+\sqrt{10}\right)}{3\left(\sqrt{x}-\sqrt{10}\right)\left(\sqrt{x}+\sqrt{10}\right)} \\ &= \lim_{x\to 10} \frac{(x-10)\left(\sqrt{x}+\sqrt{10}\right)}{3(x-10)} = \lim_{x\to 10} \frac{\sqrt{x}+\sqrt{10}}{3} = \frac{2\sqrt{10}}{3}.\end{aligned}$$

Alternate solution: We start by factoring the numerator:

$$\lim_{x\to 10} \frac{x-10}{3\left(\sqrt{x}-\sqrt{10}\right)} = \lim_{x\to 10} \frac{\left(\sqrt{x}-\sqrt{10}\right)\left(\sqrt{x}+\sqrt{10}\right)}{3\left(\sqrt{x}-\sqrt{10}\right)} = \lim_{x\to 10} \frac{\sqrt{x}+\sqrt{10}}{3} = \frac{2\sqrt{10}}{3}.$$

Any solution containing

$$\cancel{=\frac{0}{0}}, \qquad \cancel{\lim_{x\to 10} =}, \qquad \text{or} \qquad \cancel{\lim_{x\to 10} \frac{2\sqrt{10}}{3}}$$

uses incorrect notation, and will therefore lose points.

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11. (10 points) Find the exact value of the limit $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2+x-12}$. Give reasons.

Solution: This limit has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the denominator, and then cancel common factors in the fraction:

$$\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+4)} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{3+4} = \frac{1}{7}.$$

Any solution containing

$$\cancel{= \frac{0}{0}}, \qquad \cancel{\lim_{x\to 3} =}, \qquad \text{or} \qquad \cancel{\lim_{x\to 3} \frac{1}{7}}$$

uses incorrect notation, and will therefore lose points.

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12. (12 points) Use the methods of calculus to find the exact values of x at which the function $f(x) = 3x^4 - 26x^3 + 45x^2 - 24$ takes its absolute minimum and maximum on the interval $[-1, 2]$.

Hint: $f'(x) = 6x(2x-3)(x-5)$. Also, the problem does not ask you to find the exact maximum and minimum values of h on the interval, only the exact values of x at which they occur.

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we differentiate f and solve the equation $f'(x) = 0$. The derivative of f is already given for you:

$$f'(x) = 6x(2x-3)(x-5).$$

It is clearly zero when $x = 0$, $x = \frac{3}{2}$, and $x = 5$.

We now have three critical numbers, namely 0, $\frac{3}{2}$, and 5. Of these, 5 is not in the interval under consideration, so we ignore it. So we must compare the values of f at 0, $\frac{3}{2}$, and the endpoints -1 and 2. We evaluate:

$$f(-1) = 3 + 26 + 45 - 24 = 50, \qquad f(0) = -24, \qquad f\left(\tfrac{3}{2}\right) = \tfrac{75}{16} = 4.6875,$$

and

$$f(2) = 3 \cdot 16 - 26 \cdot 8 + 45 \cdot 4 - 24 = -4.$$

The smallest of these is $f(0)$ and the largest is $f(-1)$, so the absolute minimum on the interval $[-1, 2]$ occurs at $x = 0$ and the absolute maximum on the interval $[-1, 2]$ occurs at $x = -1$. Note that $x = 5$ is not correct for the minimum, even though $f(5) = -274$, because 5 is not in the interval $[-1, 2]$.

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13. (10 points/part) Differentiate the following functions:

(a) $f(y) = \left(7y^3 + \frac{1}{2}\right)\left(\frac{1}{8}y^7 - 16\sqrt{y} + \frac{2}{3}\right)$.

Solution: We use the product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Before doing so, we rewrite the factors to make then easy to differentiate:

$$f(y) = \left(7y^3 + \tfrac{1}{2}\right)\left(\tfrac{1}{8}y^7 - 16y^{1/2} + \tfrac{2}{3}\right).$$

Therefore

$$f'(y) = 21y^2\left(\tfrac{1}{8}y^7 - 16y^{1/2} + \tfrac{2}{3}\right) + \left(7y^3 + \tfrac{1}{2}\right)\left(\tfrac{7}{8}y^6 - 8y^{-1/2}\right).$$

(b) $h(t) = \dfrac{\sqrt[3]{t} - 2\pi}{\sqrt[3]{t} + 2\pi}$.

Solution: Use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Before doing so, we rewrite the numerator and denominator to make them easy to differentiate:

$$h(t) = \frac{t^{1/3} - 2\pi}{t^{1/3} + 2\pi}.$$

Therefore

$$h'(t) = \frac{\frac{1}{3}t^{-2/3}(t^{1/3} + 2\pi) - (t^{1/3} - 2\pi)\frac{1}{3}t^{-2/3}}{(t^{1/3} + 2\pi)^2} = \frac{\frac{1}{3}t^{-2/3} \cdot 4\pi}{(t^{1/3} + 2\pi)^2} = \frac{4\pi}{3t^{2/3}(t^{1/3} + 2\pi)^2}.$$

(The simplification is necessary.) Note that 2π is a *constant*, so its derivative is zero.

(c) Given that $h'(x) = 3h(x)$, find $\dfrac{d}{dx}\left(\dfrac{x}{h(x)}\right)$. (Your answer might involve the function h.)

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$\frac{d}{dx}\left(\frac{x}{h(x)}\right) = \frac{1 \cdot h(x) - xh'(x)}{h(x)^2} = \frac{h(x) - x \cdot 3h(x)}{h(x)^2} = \frac{1 - 3x}{h(x)}.$$

(The simplification is necessary.)

(d) $g(t) = \dfrac{t^2 - 27}{k + e^t} - \sqrt{17}$, where k is a constant.

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$g'(t) = \frac{2t(k + e^t) - (t^2 - 27)e^t}{\left(k + e^t\right)^2} = \frac{2kt + 2te^t - t^2e^t + 27e^t}{\left(k + e^t\right)^2}.$$

The "simplification" at the second step isn't really any simpler, so is not required. Note that $\sqrt{17}$ is a *constant*, so its derivative is zero.

(e) $q(x) = \sin(cxe^x + \pi^2)$, where c is a constant.

Solution: Use the chain rule. The product rule is needed to differentiate xe^x.

$$q'(x) = \cos(cxe^x + \pi^2) \cdot c(e^x + xe^x) = c(1 + x)e^x \cos(cxe^x + \pi^2).$$

(The rearrangement is not really necessary.) The derivative of π^2 is zero because π^2 is a constant.

(f) $f(x) = \sin\left((x^2 - k)^{17}\right)$, where k is a constant.

Solution:

$$f'(x) = \cos\left((x^2 - k)^{17}\right) \cdot 17(x^2 - k)^{16} \cdot 2x = 34x(x^2 - k)^{16} \cos\left((x^2 - k)^{17}\right).$$

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14. Let f and g be functions such that:

$$f(-3) = -5, \quad f'(-3) = 12, \quad g(-3) = 2, \qquad \text{and} \qquad g'(-3) = -3$$

and

$$f(2) = 7, \quad f'(2) = 3, \quad g(2) = -3, \qquad \text{and} \qquad g'(2) = 2.$$

Let $h(x) = f(g(x))$.

(a) (2 points) Find $h(2)$. (You will not need to use all the information provided.)

Solution: $h(2) = f(g(2)) = f(-3) = -5$.

(b) (8 points) Find $h'(2)$. (You will not need to use all the information provided.)

Solution: Using the chain rule,

$$h'(2) = f'(g(2))g'(2) = f'(-3) \cdot 2 = 12 \cdot 2 = 24.$$