SOLUTIONS TO MIDTERM 1 PROBLEMS FOR 11:00 AM

1. (1 point.) The Squeeze Theorem was invented for the purpose of torturing which of:
   (1) Functions.
   (2) Calculus students.
   (3) Professors who have to teach it in calculus classes.

Solution: I think the answer is (3).

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2. (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

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3. (a) (9 points) If $f(x) = x^2 + 5$, compute the derivative $f'(3)$ *directly from the definition of the derivative* (which you are supposed to have given above). (No credit will be given for just using the differentiation rules, but see Part (b.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 + 5 - (3^2 + 5)}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 + 5 - 9 - 5)}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \lim_{h \to 0} (6 + h) = 6. \end{aligned}$$

(b) (1 point) Use the differentiation rules we have learned to check your answer to part (a).

Solution: $f'(x) = 2x$, so $f'(3) = 6$.

Date: 31 Juanuary 2025.

4. (10 points) Differentiate the function $y(x) = (4x^3 + 8x)\sin(x) + \frac{1}{5}$. (You need not do this directly from the definition.)

Solution: The second term is a constant, so its derivative is zero. (**Don't** use the quotient rule on it!) On the first term, use the product rule:

$$\begin{aligned} y'(x) &= \frac{d}{dx}\big(4x^3 + 8x\big)\sin(x) + (4x^3 + 8x)\sin'(x) \\ &= (4 \cdot 3x^2 + 8)\sin(x) + (4x^3 + 8x)\cos(x) = (12x^2 + 8)\sin(x) + (4x^3 + 8x)\cos(x). \end{aligned}$$

Parentheses are required in the several places. For example, if

$$\cancel{\frac{d}{dx}4x^3 + 8x}, \qquad \cancel{4x^3 + 8x\cos(x)},$$

or anything similar appears in your work, it is wrong.

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5. (10 points) Differentiate the function $h(x) = \cos(3x^2 - 16x)$. (You need not do this directly from the definition.)

Solution: Use the chain rule:

$$\begin{aligned} h'(x) &= \cos'(3x^2 - 16x) \cdot \frac{d}{dx}\big(3x^2 - 16x\big) \\ &= -\sin(3x^2 - 16x) \cdot (3 \cdot 2x - 16) = -(6x - 16)\sin(3x^2 - 16x). \end{aligned}$$

Parentheses are required in the first and second steps: if

$$\cancel{\frac{d}{dx}3x^2 - 16x}, \qquad \cancel{-\sin(3x^2 - 16x) \cdot 3 \cdot 2x - 16},$$

or anything similar appears in your work, it is wrong.

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6. (10 points) Find the exact value of the limit $\lim_{x \to -4} \frac{x^2 - 16}{x^2 + x - 12}$, or explain why this limit does not exist.

Solution: This limit has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the numerator and denominator, cancel common factors in the fraction, and then use the limit laws:

$$\lim_{x \to -4} \frac{x^2 - 16}{x^2 + x - 12} = \lim_{x \to -4} \frac{(x-4)(x+4)}{(x-3)(x+4)} = \lim_{x \to -4} \frac{x-4}{x-3} = \frac{-4-4}{-4-3} = \frac{8}{7}.$$

Any solution containing

$$\cancel{=\frac{0}{0}}, \qquad \cancel{\lim_{x \to -4} =}, \qquad \text{or} \qquad \cancel{\lim_{x \to -4} \frac{8}{7}}$$

uses incorrect notation, and will therefore lose points.

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7. (20 points.) An eccentric gardener wants to build a rectangular walled enclosure. The north and south walls will cost 2 pesos per foot, the west wall will cost 5 pesos per foot, and the east

wall will cost 3 pesos per foot. The gardener has 800 pesos available to build the enclosure. Use calculus to find the lengths of the south and west walls of the largest such enclosure that can be built. Include units. Verify that the solution corresponds to the largest possible area, using methods we have seen so far in the course (not the second derivative test).

Solution: Let x be the length of the south and north walls (in feet), and let y be the length of the west and east walls (in feet). Let A be the area. Thus

$$A = xy. \tag{1}$$

We want to maximize this quantity. Let C be the total cost, in pesos.

We have

$$C = 2x + 5y + 2x + 3y.$$

(The terms are, in order, the costs of the north, west, south, and east walls.) We are also given $C = 800$. Combining the two formulas for C:

$$800 = C = 2x + 5y + 2x + 3y = 4x + 8y.$$

Solve for one of the variables, say x:

$$x = \frac{1}{4}(800 - 8y) = 200 - 2y. \tag{2}$$

Substitute this in (1) and write as a function of y:

$$A(y) = (200 - 2y)y = 200y - 2y^2.$$

We obviously must have $y \geq 0$. Also, clearly $x \geq 0$. By (2), this means $200 - 2y \geq 0$, so $y \leq 100$. Therefore we need to *maximize* $A(y) = 200y - 2y^2$ on the interval $[0, 100]$.

Find the critical points: $A'(y) = 200 - 4y$, so $A'(y) = 0$ exactly when $y = 50$. Now compare the values of A at the critical points and endpoints:

$$A(50) = (200 - 2 \cdot 50) \cdot 50 = 15{,}000, \qquad A(0) = 0, \qquad \text{and} \qquad A(100) = 0.$$

The largest is clearly $A(50)$. So take $y = 50$. By (2), this means $x = 100$. So the largest area is gotten by taking the south wall to be $x = 100$ feet long and the west wall to be $y = 50$ feet long.

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8. (13 points) Use the methods of calculus to find the exact values of x at which the function $f(x) = x^3 - 6x^2 + 16$ takes its absolute minimum and maximum on the interval $[-1, 2]$.

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers in the interval and at the endpoints, and compare values.

To find the critical numbers, we differentiate f, solve the equation $f'(x) = 0$, and find all numbers x in our interval such that $f'(x)$ does not exist. The derivative of f is

$$f'(x) = 3x^2 - 12x = 3x(x - 4).$$

It exists everywhere, and it is zero when $x = 0$ and $x = 4$.

Since $x = 4$ is not in $[-1, 2]$, we ignore it. (**I must see you reject** 4. If I don't see this, I will assume you didn't correctly solve the equation $f'(x) = 0$.)

We now have one critical number, namely 0. So we must compare the values of f at 0, and at the endpoints -1 and 2.

Since $f(-1) = 9$, $f(0) = 16$, and $f(2) = 0$, the smallest of these is $f(2)$ and the largest is $f(0)$, So the absolute minimum on the interval $[-1, 2]$ occurs at $x = 0$ and the absolute maximum on the interval $[-1, 2]$ occurs at $x = 2$.

Note that $x = 4$ is not correct for the minimum, even though $f(4) = -16$, because 4 is not in the interval $[-1, 2]$.

No credit will be given for any solution which does not show evidence of an attempt to find the critical numbers of f. In particular, no credit will be given for simply comparing the values of f at the integers in the interval.

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9. (10 points.) This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\lim_{x\to-5} \dfrac{x^3 + 5x^2 + 4x + 20}{x + 5}$. The method is to factor the numerator and cancel one of the factors. The factors of the numerator are $x + 5$ and $x^2 + 4$.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. In particular, put "=" and "lim" everywhere they belong, and nowhere else. Start by writing $\lim_{x\to-5} \dfrac{x^3 + 5x^2 + 4x + 20}{x + 5}$. Show at least the following steps:

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x = -5$.
(3) After substituting $x = -5$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified.

Solution.

$$\begin{aligned}\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5} &= \lim_{x\to-5} \frac{(x+5)(x^2+4)}{x+5} \\ &= \lim_{x\to-5}(x^2+4) = (-5)^2 + 4 = 25 + 4 = 29.\end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq -5$, we have

$$\frac{x^3 + 5x^2 + 4x + 20}{x + 5} = \frac{(x+5)(x^2+4)}{x+5} = x^2 + 4.$$

Therefore

$$\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5} = \lim_{x\to-5}(x^2+4) = (-5)^2 + 4 = 25 + 4 = 29.$$

This completes the solution. □

Comments. In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

The symbol "$\lim_{x\to-5}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\lim_{x\to-5}(x^2+4) = \lim_{x\to-5}[(-5)^2+4]$$

is a mathematically true statement, but does not correctly show the intended step. Also, ~~-5^2~~ is wrong: $-5^2 = -25$, but what is wanted is $(-5)^2 = 25$.

Every parenthesis shown is essential. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.

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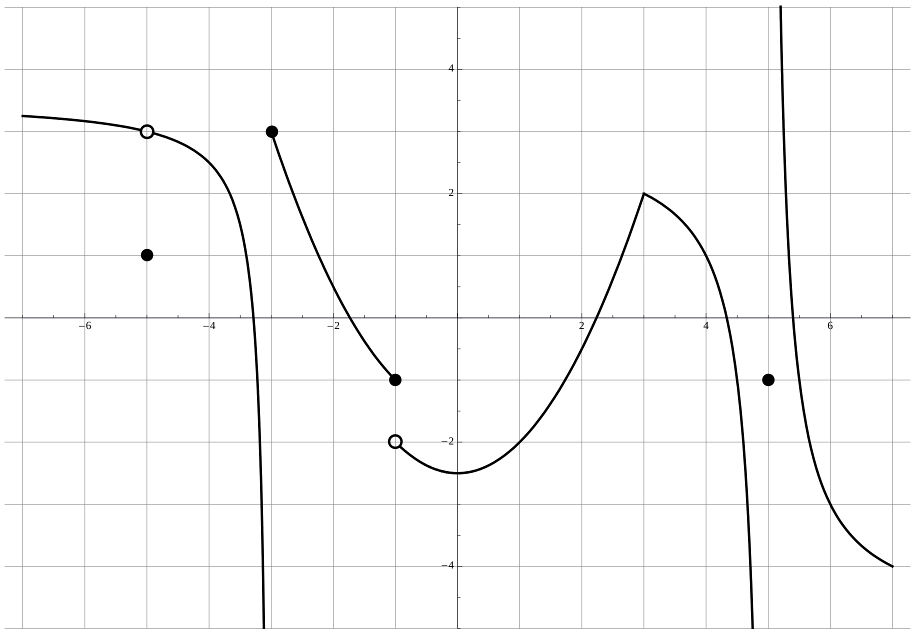
10. (5 points/part) For the function $y = g(x)$ graphed below, answer the following questions:



(a) Which of the following best describes $g'(4)$? Why?

(1) $g'(4)$ does not exist.
(2) $g'(4)$ is close to 0.
(3) $g'(4)$ is positive and not close to 0.
(4) $g'(4)$ is negative and not close to 0.
(5) None of the above.

Solution: $g'(4)$ is the slope of the tangent line to the graph of $y = g(x)$ at $x = 4$. You can tell from inspection that this slope is negative and not close to 0 (choice (4) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around -2. (In fact, it is clear that $g'(x) < 0$ when $3 < x < 5$.)

(b) Does $\lim_{x \to -5} g(x)$ exist? If so, what is it? If not, why not?

Solution: You can see from the graph that one can make $g(x)$ as close as one wants to 3 by requiring that x be close enough to -5 but different from -5. Therefore $\lim_{x \to -5} g(x) = 3$.

It is not true that ~~$\lim_{x \to -5} g(x)$~~ $= 1$. That is $g(-5)$.

Extra credit. (15 extra credit points; grading will be harsher than on related problems on the main exam. Do not attempt this problem until you have done and checked your answer to all the ordinary problems on this exam. It will only be counted if you get a grade of B or better on the main part of this exam.)

Let f be a function such that $f'(t) = \sqrt[3]{7 + \arctan(t)}$. Let

$$g(x) = \sin\left(\left[\left([f(x)+7]^{16}+11\right)^{1/12}+e^{2x}\right]^{99}\right).$$

Find $g'(x)$.

Solution (sketchier than for related problems on the main exam): We use the chain rule a number of times, getting

$$\begin{aligned} g'(x) &= \cos\left(\left[\left([f(x)+7]^{16}+11\right)^{1/12}+e^{2x}\right]^{99}\right)\cdot 99\left[\left([f(x)+7]^{16}+11\right)^{1/12}+e^{2x}\right]^{98} \\ &\quad\cdot\left[\frac{1}{12}\left([f(x)+7]^{16}+11\right)^{-11/12}\cdot 16\,[f(x)+7]^{15}\cdot f'(x)+2e^{2x}\right] \\ &= 99\cos\left(\left[\left([f(x)+7]^{16}+11\right)^{1/12}+e^{2x}\right]^{99}\right)\left[\left([f(x)+7]^{16}+11\right)^{1/12}+e^{2x}\right]^{98} \\ &\quad\cdot\left[\frac{4}{3}\left([f(x)+7]^{16}+11\right)^{-11/12}[f(x)+7]^{15}\,\sqrt[3]{7+\arctan(x)}+2e^{2x}\right]. \end{aligned}$$