Math 251 Final Exam NAME: SOLUTIONS

Th 7 Dec. 2017 Student ID: $\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$

Reminder: Exams will be available for inspection when they are graded, probably by Monday 11 Dec. I keep the originals, but you can get a copy on request. Grading complaints must be submitted in writing before final grades are turned in.

Extra credit for catching errors is available through 5:00 pm Sunday 10 Dec.

1. (10 points/part.) Find the exact values of the following limits (possibly including $\infty$ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give justification in all cases (not just heuristic arguments). Remember to use correct notation.

(a) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{\cos(x)-3}$.

*Solution:* As always, the first thing we do is try to substitute $x = 3$. We get

$$\lim_{x\to 3} \frac{x^2-9}{\cos(x)-3} = \frac{3^2-9}{\cos(3)-3} = \frac{0}{\cos(3)-3} = 0.$$

Note that L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\lim_{x\to 3} \frac{2x}{-\sin(x)} = -\frac{6}{\sin(3)},$$

which is the wrong answer.

(b) $\displaystyle\lim_{y\to\infty} \frac{2y+216}{13y-5\sin(y)}$. (Be sure to show your work!)

*Solution:* The limit has the indeterminate form "$\frac{\infty}{\infty}$". Therefore more work is needed. We factor out $y$ from both the numerator and denominator, and then use the limit laws:

$$\lim_{y\to\infty} \frac{2y+216}{13y-5\sin(y)} = \lim_{y\to\infty} \frac{2+\frac{216}{y}}{13-\frac{5\sin(y)}{y}} = \frac{2+\lim_{y\to\infty}\frac{216}{y}}{13-\lim_{y\to\infty}\frac{5\sin(y)}{y}}.$$

Certainly $\lim_{y\to\infty} \frac{216}{y} = 0$. Since $-5 \leq 5\sin(y) \leq 5$ for all $y$, we get $\lim_{y\to\infty} \frac{5\sin(y)}{y} = 0$ by the Squeeze Theorem. Therefore

$$\lim_{y\to\infty} \frac{2y+216}{13y-5\sin(y)} = \frac{2}{13}.$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{y\to\infty} \frac{2y+216}{13y-5\sin(y)} = \lim_{y\to\infty} \frac{\left(\frac{1}{y}\right)(2y+216)}{\left(\frac{1}{y}\right)(13y-5\sin(y))} = \lim_{y\to\infty} \frac{2+\frac{216}{y}}{13-\frac{5\sin(y)}{y}}$$
$$= \frac{2+\lim_{y\to\infty}\frac{216}{y}}{13-\lim_{y\to\infty}\frac{5\sin(y)}{y}} = \frac{2+0}{13+0} = \frac{2}{13}.$$

The hypotheses of L'Hospital's Rule are satisfied, but it doesn't help. Applying it once gives

$$\lim_{y\to\infty} \frac{2}{13+2\cos(y)}.$$

This limit does not exist: the function oscillates between $\frac{2}{15}$ and $\frac{2}{9}$.

(c) $\displaystyle\lim_{x\to 0} \frac{5x^2}{1-\cos(7x)}$.

*Solution:* The limit has the indeterminate form $\dfrac{0}{0}$. Therefore we may try L'Hospital's Rule. The chain rule gives $\dfrac{d}{dx}\big(1-\cos(7x)\big) = 7\sin(7x)$, so

$$\lim_{x\to 0} \frac{5x^2}{1-\cos(7x)} = \lim_{x\to 0} \frac{10x}{7\sin(7x)}$$

if the second limit exists. The second limit also has the indeterminate form $\dfrac{0}{0}$. Therefore we may try L'Hospital's Rule again. A similar calculation shows that

$$\lim_{x\to 0} \frac{10x}{7\sin(7x)} = \lim_{x\to 0} \frac{10}{49\cos(7x)},$$

if the limit on the right exists. But the limit on the right is equal to $10/[49\cos(7\cdot 0)] = 10/49$. Therefore

$$\lim_{x\to 0} \frac{5x^2}{1-\cos(7x)} = \frac{10}{49}.$$

(d) $\displaystyle\lim_{x\to 2^-} \frac{e^{-x}}{x-2}$. (Be sure to show your work!)

*Solution:* At 2, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at 2. Therefore the function $f(x) = \frac{e^{-x}}{x-2}$ has a vertical asymptote at $x = 2$.

For $x < 2$ but very close to 2, the denominator is negative and very close to zero. The numerator is very close to $e^{-2} = \frac{1}{e^2} > 0$, so is positive and not close to zero. Therefore the quotient is negative and very far from zero. Thus

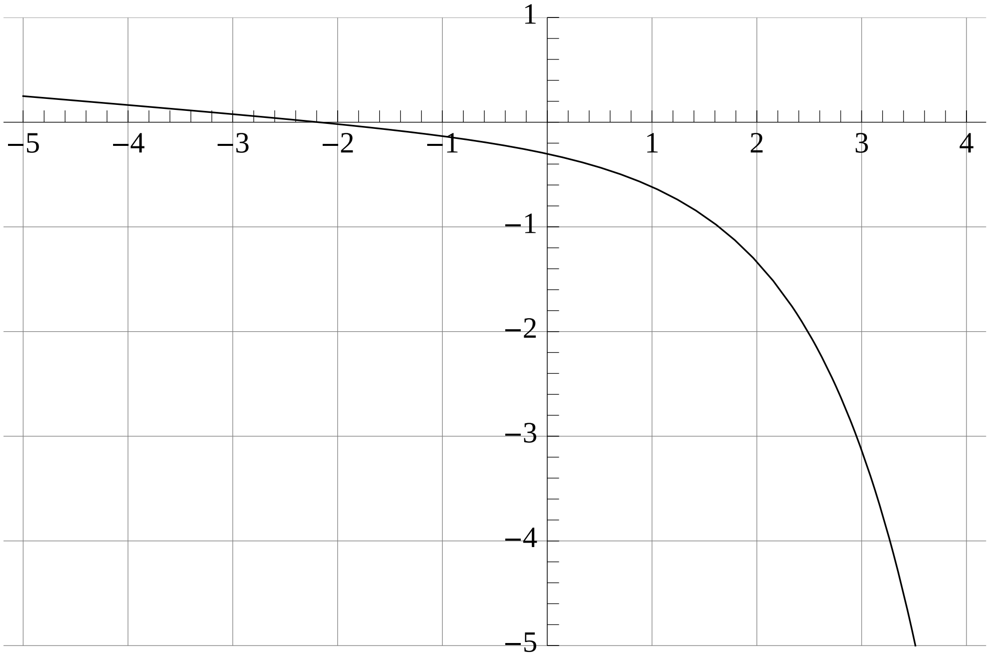
$$\lim_{x\to 2^-} \frac{e^{-x}}{x-2} = -\infty.$$

L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

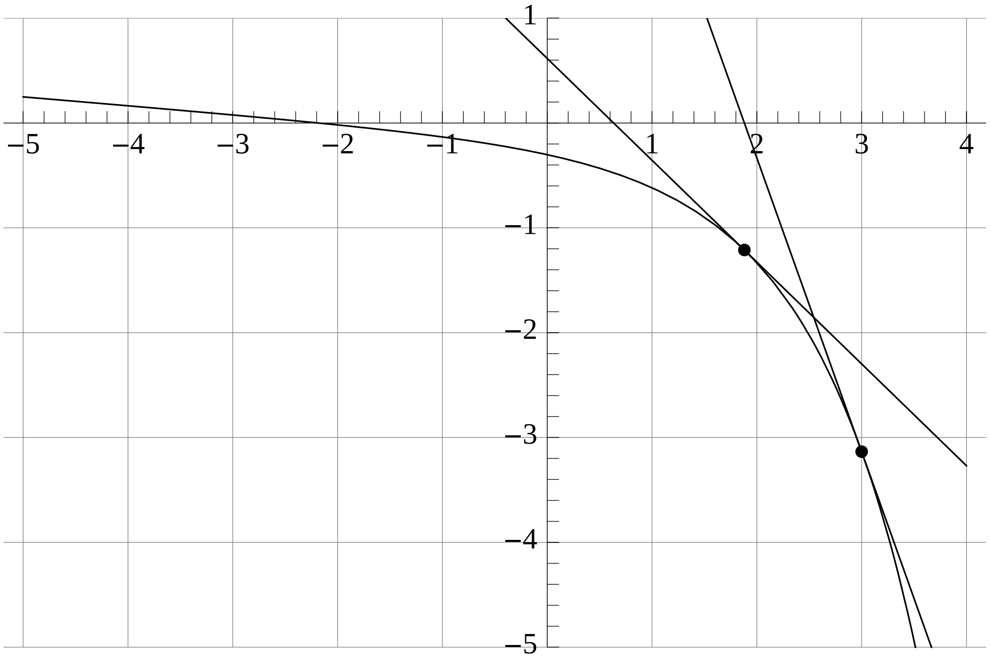
$$\lim_{x\to 2^-} \frac{-e^{-x}}{1} = -\frac{1}{e^2},$$

which is the wrong answer.

2. (10 points.) The picture below shows the graph of a function $f$. Suppose that Newton's method is used to approximate the root of the equation $f(x) = 0$ which is visible in the picture, with initial approximation $x_1 = 3$. Draw on the graph the tangent lines that are used to find $x_2$ and $x_3$, and estimate the numerical values of $x_2$ and $x_3$.




*Solution:* Here is a graph showing the correct tangent lines. The two dots mark the points on the graph at which they are tangent.




The values of $x_2$ and $x_3$ are $x_2 \approx 1.9$ and $x_3 \approx 0.6$. Since it is hard to draw accurate tangent lines on a graph, answers do not need to be this accurate to be considered correct.

For those interested, the function has the formula $f(x) = -\frac{1}{6} - e^{-x-2} - \frac{1}{12}x$, and the slope of tangent line at $x = x_1$ is

$$-\tfrac{1}{12} - e \approx -2.80162,$$

giving

$$x_2 = \frac{24e - 2}{1 + 12e} \approx 1.88102.$$

The slope of tangent line at $x = x_2$ is approximately $-0.97116$, giving $x_3 \approx 0.633806$. (The exact expressions are too messy to reproduce here.) The root is approximately $-2.18303$.

3. (12 points) Find the equation of tangent line to the graph of $g(x) = 2x + 4\sqrt{3x-2}$ at $x = 2$. You need not calculate the derivative directly from the definition.

*Solution:* We need the slope, which is $g'(2)$, and a point on the line, such as

$$(2,\ g(2)) = \left(2,\ 2 \cdot 2 + 4\sqrt{3 \cdot 2 - 2}\right) = \left(2,\ 4 + 4\sqrt{4}\right) = (2, 12).$$

To find $g'(2)$, we rewrite the function as $g(x) = 2x + 4(3x-2)^{1/2}$. Using the chain rule on the second term, we get

$$g'(x) = 2 + 4\left(\frac{1}{2} \cdot (3x-2)^{-1/2} \cdot 3\right) = 2 + \frac{6}{\sqrt{3x-2}},$$

so

$$g'(2) = 2 + \frac{6}{\sqrt{3 \cdot 2 - 2}} = 2 + \frac{6}{2} = 5.$$

Therefore the equation of the tangent line is $y - 12 = 5(x-2)$, which can be rearranged to give $y = 5x + 2$.

We want the slope at the *particular* value $x = 2$. Therefore we must substitute $x = 2$ in the formula for the derivative $g'(x)$ *before* using it as the slope of a line. The equation

$$y - 12 = \left(2 + \frac{6}{\sqrt{3x-2}}\right)(x-2)$$

is wrong—it is not even the equation of any line.

4. (10 points/part) Differentiate the functions as requested.

(a) Find $f'(x)$, where $f(x) = \pi^3 + \dfrac{2x+1}{x^2+1}$.

*Solution:* The derivative of $\pi^3$ is zero because $\pi^3$ is a constant. On the rest, use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

This gives

$$f'(x) = \frac{2 \cdot (x^2+1) - (2x+1) \cdot (2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2} = \frac{-2x^2 - 2x + 2}{(x^2+1)^2}.$$

(The simplification is required.)

(b) Let $f(t) = e^{7t + \arcsin(t)} + \csc\left(\frac{\pi}{3}\right)$. Find $f'(t)$.

*Solution:* We use the chain rule. The derivative of $\csc\left(\frac{\pi}{3}\right)$ is zero because $\csc\left(\frac{\pi}{3}\right)$ is a constant. Thus

$$f'(t) = e^{7t + \arcsin(t)} \cdot \frac{d}{dt}\big(7t + \arcsin(t)\big) = e^{7t + \arcsin(t)}\left(7 + \frac{1}{\sqrt{1-t^2}}\right).$$

One can rewrite this expression as

$$f'(t) = 7e^{7t + \arcsin(t)} + \frac{e^{7t + \arcsin(t)}}{\sqrt{1-t^2}},$$

but this isn't really an improvement, and is not required.

(c) Let $q(x) = \ln(x)\cos(x^2 - cx)$, where $c$ is a constant. Find $q'(x)$.

*Solution:* Use the product rule. The chain rule is needed to differentiate $\cos(x^2 - cx)$. Thus:

$$\begin{aligned} q'(x) &= \frac{d}{dx}\big(\ln(x)\big)\cos(x^2 - cx) + \ln(x)\frac{d}{dx}\big(\cos(x^2 - cx)\big) \\ &= \frac{1}{x}\cos(x^2 - cx) + \ln(x)(-\sin(x^2 - cx))(2x - c) \\ &= \frac{1}{x}\cos(x^2 - cx) - \ln(x)(2x - c)\sin(x^2 - cx). \end{aligned}$$

5. (30 points.) A 5 meter ladder leans against a vertical wall in a room with a high ceiling and level floor. Because the floor is slippery, the foot of the ladder is sliding away from the wall. When the foot is 3 meters from the wall, it is sliding away at 7 meters per hour. At this time, how fast is the top of the ladder sliding down the wall? (Be sure to include correct units.)

*Solution 1:* Note: There is no picture in this file.

Let $x(t)$ be the distance of the foot of the ladder from the base of the wall, in meters. Let $h(t)$ be the height above the floor of the top of the ladder, also in meters. Recall that the ladder is 5 meters long, and this length does not change. The missing picture should therefore show a right triangle, with horizontal side labelled $x(t)$, vertical side labelled $h(t)$, and hypotenuse (diagonal) labelled 5.

Let $t_0$ be the time at which we are interested. We are given the following:

$$x(t_0) = 3 \quad \text{and} \quad x'(t_0) = 7.$$

(The derivative $x'(t_0)$ is positive because the distance from the foot of the ladder to the wall is increasing.)

Our quantities are related by the equation $[x(t)]^2 + [h(t)]^2 = 5^2$, since we have a right triangle. Differentiating this equation with respect to $t$ gives

$$2x(t)x'(t) + 2h(t)h'(t) = 0.$$

(Don't forget to use the chain rule!) Dividing through by 2 and substituting $t_0$ for $t$ gives

$$x(t_0)x'(t_0) + h(t_0)h'(t_0) = 0.$$

Substituting for the known quantities gives

$$3 \cdot 7 + h(t_0)h'(t_0) = 0.$$

We want to find $h'(t_0)$, so we still need $h(t_0)$. Substituting for the known quantities in the equation $[x(t_0)]^2 + [h(t_0)]^2 = 5^2$ gives $3^2 + [h(t_0)]^2 = 5^2$, which can be solved to give $h(t_0) = 4$. (We reject the solution $h(t_0) = -4$, because $h(t)$ is positive.) Therefore $3 \cdot 7 + 4h'(t_0) = 0$, whence

$$h'(t_0) = -\frac{3 \cdot 7}{4} = -\frac{21}{4}.$$

So the top of the ladder is sliding down the wall at $\frac{21}{4}$ meters per hour. (The units are required.)

*Solution 2:* Draw the same picture as in Solution 1, and label it the same way. Then obtain the equation $[x(t)]^2 + [h(t)]^2 = 5^2$ as before. Rewrite this as

$$h(t) = \sqrt{5^2 - [x(t)]^2}.$$

(We use the positive square root, because $h(t)$ is positive.) Differentiating this equation with respect to $t$ gives

$$h'(t) = \tfrac{1}{2}\left(5^2 - [x(t)]^2\right)^{-1/2} \cdot (-2x(t)x'(t)) = -\frac{x(t)x'(t)}{\sqrt{5^2 - [x(t)]^2}}.$$

(Don't forget to use the chain rule!) Setting $t = t_0$ and substituting for the known quantities gives

$$h'(t_0) = -\frac{3 \cdot 7}{\sqrt{5^2 - 3^2}} = -\frac{21}{4}.$$

So the top of the ladder is sliding down the wall at $\frac{21}{4}$ meters per hour. (The units are required.)

*Solution 3:* Here, for reference, is what the first solution looks like in physicists' notation.

We have $x^2 + h^2 = 5^2$. Differentiate with respect to $t$:

$$2x\frac{dx}{dt} + 2h\frac{dx}{dt} = 0.$$

(Don't forget the factors $\frac{dx}{dt}$ and $\frac{dh}{dt}$!) Use the equation $x^2 + h^2 = 5^2$ and the fact that $x = 3$ at the time of interest to find that $h = 4$ at the time of interest. Then substitute $x = 3$, $h = 4$, and $\frac{dx}{dt} = 7$, getting

$$3 \cdot 7 + 4\frac{dh}{dt} = 0,$$

and solve for $\frac{dh}{dt}$ the same way we solved for $h'(t_0)$ above.

6. A fire-breathing monster is thrown upwards on the planet Yuggxth. Its height $t$ seconds after it is thrown is $40t - 5t^2$ feet, until it hits the ground again.

(a) (4 points) Is the monster falling or rising 6 seconds after being thrown? How fast?

*Solution:* Let $y(t)$ be the height, in feet, of the monster $t$ seconds after it is thrown. Then $y(t) = 40t - 5t^2$ for $t$ at most the time at which it hits the ground. Since $y(6) = (40)(6) - (6)(5^2) = 240 - 150 > 0$, after 6 seconds it hasn't hit the ground yet. Therefore the vertical velocity at time $t$ is $y'(t) = 40 - 10t$, and the vertical velocity at time 6 is $y'(6) = 40 - (10)(6) = -20$. Therefore the monster is falling at 20 feet/second.

Note: You *must* include the units in this kind of problem.

It is not correct to say that the monster is ~~falling at $-20$~~ feet/second. That would mean it is rising at 20 feet/second.

(b) (7 points) How long after being thrown does the monster reach its maximum height?

*Solution:* Let $y(t)$ be the height, in feet, of the monster $t$ seconds after it is thrown. Then $y(t) = 40t - 5t^2$ for $t$ at most the time at which it hits the ground. The monster reaches its greatest height when it stops rising and starts falling, that is, when the derivative $y'(t)$ changes from positive to negative. We have $y'(t) = 40 - 10t$, which changes from positive to negative when it is zero, that is, at $t = 4$. So it reaches its greatest height 4 seconds after being thrown.

Note: You *must* include the units in this kind of problem.

The graph of position as a function of time is a parabola which opens downward. The high point is therefore at the vertex of the parabola. There are ways to find the vertex without explicitly using calculus, and these are correct solutions. However, any such solution must explicitly state that the vertex of the parabola is being found, and then use a mathematically correct way to do so; otherwise, it will receive no credit. (When I read a solution which uses a formula to find the vertex of the parabola but gives no explanation for why doing so is relevant, I see no relation to the question being asked.)

7. (15 points) If $y^3 = \sin(11x - y) - \sin(7)$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

*Solution:* Let's write it with $y$ as an explicit function $y(x)$ of $x$:

$$y(x)^3 = \sin(11x - y(x)) - \sin(7).$$

Differentiate both sides with respect to $x$, using the chain rule on both sides:

$$3[y(x)]^2 y'(x) = \cos(11x - y(x))\frac{d}{dx}(11x - y(x)) = \cos(11x - y(x))\left(11 - y'(x)\right).$$

(The derivative of $\sin(7)$ is zero because $\sin(7)$ is a constant.)

Now solve for $y'(x)$:

$$3[y(x)]^2 y'(x) = 11\cos(11x - y(x)) - \cos(11x - y(x))y'(x)$$

$$3[y(x)]^2 y'(x) + \cos(11x - y(x))y'(x) = 11\cos(11x - y(x))$$

$$y'(x) = \frac{11\cos(11x - y(x))}{3[y(x)]^2 + \cos(11x - y(x))}.$$

This expression can't be further simplified.

For those who prefer the other notation, here it is written with $\frac{dy}{dx}$. Differentiate with respect to $x$, using the chain rule on both sides, just as before:

$$3y^2\frac{dy}{dx} = \cos(11x - y)\frac{d}{dx}(11x - y) = \cos(11x - y)\left(11 - \frac{dy}{dx}\right).$$

(The derivative of $\sin(7)$ is zero because $\sin(7)$ is a constant.)

Now solve for $\frac{dy}{dx}$:

$$3y^2\frac{dy}{dx} = 11\cos(11x - y) - \cos(11x - y)\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} + \cos(11x - y)\frac{dy}{dx} = 11\cos(11x - y)$$

$$\frac{dy}{dx} = \frac{11\cos(11x - y)}{3y^2 + \cos(11x - y)}.$$

As before, this expression can't be further simplified.

8. (17 points) Suppose we know the following about the function $h$:

(1) $h$ is defined and continuous on $(-\infty, \infty)$, and $h'(x)$ and $h''(x)$ exist on all of $(-\infty, \infty)$.
(2) $h$ has only one critical number, namely 1.
(3) $h(1) \approx -2.718$.
(4) $h'(x) < 0$ for $x$ in the interval $(-\infty, 1)$.
(5) $h'(x) > 0$ for $x$ in the interval $(1, \infty)$.
(6) The only solution to $h''(x) = 0$ is $x = 0$.
(7) $h(0) = -2$.
(8) $h''(x) < 0$ for $x$ in the interval $(-\infty, 0)$.
(9) $h''(x) > 0$ for $x$ in the interval $(0, \infty)$.
(10) $\lim_{x\to-\infty} h(x) = 0$.

Find the asymptotes, intervals of increase and decrease, local minimums and maximums, intervals of concavity up and down, and inflection points. Then draw the graph of $h$. Make sure that the graph matches the information about concavity etc. that you found.

*Solution:* By (4) and (5), the function $h$ is strictly decreasing on $(-\infty, 1)$ and strictly increasing on $(1, \infty)$. Therefore $h$ has a local minimum at $x = 1$. This is the only possible location for a local minimum or a local maximum, because 1 is the only critical number.

By (8) and (9), the function $h$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Therefore $h$ has an inflection point at $x = 0$.

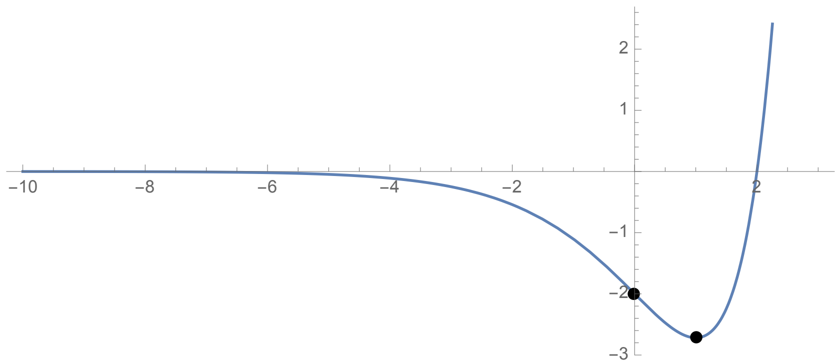
By (9), the function $h$ has a horizontal asymptote at $y = 0$, which is approached in the negative direction.

To get the graph, it is perhaps easiest to break up the intervals and behavior at all the points listed above, as follows.

| Interval | Behavior of $h$ |
|---|---|
| $(-\infty, 0)$ | Strictly decreasing and concave down |
| $(0, 1)$ | Strictly decreasing and concave up |
| $(1, \infty)$ | Strictly increasing and concave up |

Since $h$ is strictly increasing and concave up on $(1, \infty)$, the rate at which $h(x)$ increases must increase with $x$. Therefore $\lim_{x\to-\infty} h(x) = \infty$. So no horizontal asymptote can be approached in the positive direction. There are no vertical asymptotes since $h$ is defined and continuous on the whole real line.

Here is the graph of a function satisfying these conditions. The points in (3) and (7) are shown with black dots.




**Note:** A correct graph must clearly show that $h(x)$ goes to $+\infty$ as $x \to \infty$.

In case anyone is interested, the function I used is $h(x) = (x-2)e^x$. Its first two derivatives are

$$h'(x) = e^x + (x-2)e^x = (x-1)e^x \quad \text{and} \quad h''(x) = e^x + (x-1)e^x = e^x.$$

9. (35 points.) You have a magical animal which is much stronger when going north than in any other direction. You want to construct a rectangular enclosure for this animal. The fence on the north side costs 6 florins per yard, and the fence on the south, west, and east sides costs 2 florins per yard. The area of the enclosure is to be 200 square yards. What are the dimensions of the cheapest possible enclosure?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

*Solution:* Note: There is no picture in this file.

The arrangement of fences will be a rectangle with sides running north-south and east-west, with area 200 square yards. We are supposed to minimize the total cost of fence; call it $C$.

Let $x$ be the length of the enclosure in the east-west direction, and let $y$ be its length in the north-south direction, both measured in yards.

We must determine the total cost of fence. The fence on the north side costs $6x$ florins, the fence on the south side costs $2x$ florins, the fence on the east side costs $2y$ florins, the fence on the west side costs $2y$ florins. Adding these together give a total cost $C$ of

$$c = 6x + 2x + 2y + 2y = 8x + 4y.$$

We are to minimize this expression. It has too many variables, and ignores the fact that $x$ and $y$ are not independent. We must eliminate one of the variables by using the requirement on the area. The area is $A = xy$ (in square yards), which is supposed to be 200. It doesn't matter much which variable you solve for. I decided to solve for $y$, giving

$$y = \frac{200}{x}.$$

Substitute this for $y$ in the formula $L = 8x + 4y$, write it as a function of $x$, and rearrange it for convenience:

$$C(x) = 4\left(\frac{200}{x}\right) + 8x = \frac{800}{x} + 8x = 800x^{-1} + 8x.$$

The constraints are that $x$ and $y$ are both strictly positive. Since $y = 200x^{-1}$, positivity of $y$ follows from positivity of $x$. Our problem is therefore to minimize the function $C(x) = 800x^{-1} + 8x$ for $x$ in the interval $(0, \infty)$.

We search for critical numbers. Differentiate:

$$C'(x) = -800x^{-2} + 8.$$

Set the derivative equal to zero and solve:

$$0 = C'(x) = -800x^{-2} + 8 = \frac{-800 + 8x^2}{x^2}.$$

This can be zero only when the numerator is zero, that is, when $8x^2 = 800$. This happens if $x^2 = 100$, that is, $x = 10$ or $-10$. We reject $x = -10$ because it is not in the interval $(0, \infty)$. (I must see you do this! Otherwise, I don't know that you correctly solved the equation $C'(x) = 0$.) So 10 is the only critical number in the interval $(0, \infty)$.

Is it a maximum or minimum? The easiest test to use here is the second derivative test. One easily checks that

$$C''(x) = (-2)(-800)x^{-3} = 1600x^{-3},$$

which is positive for all $x$ in the interval $(0, \infty)$. Therefore $C$ is concave up everywhere on the interval $(0, \infty)$, and any critical number must be an absolute minimum.

The problem asked for the dimensions giving the minimum cost. The lengths of the north and south sides are 10 yards. Since $y = 200x^{-1}$, the east and west sides have length $200/10 = 20$ yards. (Don't forget the units!)

For those who tried it instead of the second derivative test as done above, here is how the first derivative test works. We factor the derivative as follows:

$$C'(x) = 8x^{-2}(x^2 - 100).$$

All factors except $x^2 - 100$ are always positive. For $0 < x < 10$, the factor $x^2 - 100$ is negative. Thus $C'(x) < 0$, and $C$ is decreasing on the whole interval $(0, 10)$. For $x > 10$, the factor $x^2 - 100$ is positive. Thus $C'(x) > 0$, and $C$ is increasing on the whole interval $(10, \infty)$. Clearly, then, there is an absolute minimum at $x = 10$.

Here is a different approach to the first derivative test. We know that $C'(x)$ is continuous on the whole interval $(0, \infty)$. It is zero only at $x = 10$. Therefore it must have the same sign throughout the interval $(0, 10)$, and we can find that sign by computing, say, $C'(1) = -799 < 0$. The derivative must also have the same sign throughout the interval $(10, \infty)$, and we get it by computing, say, $C'(11) = 11^{-2} \cdot (11^2 - 10^2) > 0$. As above, then, $C(x)$ is decreasing on the whole interval $(0, 10)$ and increasing on the whole interval $(10, \infty)$, so has an absolute minimum at $x = 10$.

Finally, here is the test using limits at the ends of the domain. We compare $C(10) = 800/10 + 8 \cdot 10 = 400 + 400 = 800$,

$$\lim_{x \to 0^+} C(x) = \lim_{x \to 0^+} \left( \frac{800}{x} + x \right) = \infty,$$

and

$$\lim_{x \to \infty} C(x) = \lim_{x \to \infty} \left( \frac{800}{x} + x \right) = \infty.$$

Obviously 800 is the smallest of these.

EC1. (15 extra credit points; grading will be harsher than on related problems on the main exam.) Let $f$ be a function such that $f'(t) = \sqrt[3]{7 + \arctan(t)}$. Let

$$g(x) = \sin\left( \left[ \left( [f(x) + 7]^{16} + 11 \right)^{1/12} + e^{2x} \right]^{99} \right).$$

Find $g'(x)$.

*Solution* (sketchier than for related problems on the main exam): We use the chain rule a number of times, getting

$$\begin{aligned} g'(x) &= \cos\left( \left[ \left( [f(x) + 7]^{16} + 11 \right)^{1/12} + e^{2x} \right]^{99} \right) \cdot 99 \left[ \left( [f(x) + 7]^{16} + 11 \right)^{1/12} + e^{2x} \right]^{98} \\ &\quad \cdot \left[ \frac{1}{12} \left( [f(x) + 7]^{16} + 11 \right)^{-11/12} \cdot 16 \, [f(x) + 7]^{15} \cdot f'(x) + 2e^{2x} \right] \\ &= 99 \cos\left( \left[ \left( [f(x) + 7]^{16} + 11 \right)^{1/12} + e^{2x} \right]^{99} \right) \left[ \left( [f(x) + 7]^{16} + 11 \right)^{1/12} + e^{2x} \right]^{98} \\ &\quad \cdot \left[ \frac{4}{3} \left( [f(x) + 7]^{16} + 11 \right)^{-11/12} [f(x) + 7]^{15} \sqrt[3]{7 + \arctan(x)} + 2e^{2x} \right]. \end{aligned}$$

EC2. (15 extra credit points) Find a real number $r$ such that

$$\lim_{x \to 0} \frac{\sin(x) - (x + rx^3)}{x^5}$$

exists (in particular, is not infinite), and for this choice of $r$ find the limit above.

*Solution:* We use L'Hospital's Rule three times, assuming that it doesn't convert a limit which exists into one which doesn't exist. With each application, since the denominator approaches zero as $x \to 0$, if the limit is to be finite then the numerator must also approach zero as $x \to 0$. That is, we must actually have an indeterminate form. Thus:

$$\lim_{x\to 0} \frac{\sin(x) - (x + rx^3)}{x^5} = \lim_{x\to 0} \frac{\cos(x) - (1 + 3rx^2)}{5x^4} = \lim_{x\to 0} \frac{-\sin(x) - 6rx}{20x^3}$$
$$= \lim_{x\to 0} \frac{-\cos(x) - 6r}{60x^2}.$$

Since $\lim_{x\to 0} \cos(x) = 1$ and $\lim_{x\to 0}(-\cos(x) - 6r)$ must be zero (otherwise, the last limit will fail to exist), it follows that $r = -\frac{1}{6}$.

Now we can continue using L'Hospital's Rule two more times:

$$\lim_{x\to 0} \frac{-\cos(x) - 6r}{60x^2} = \lim_{x\to 0} \frac{\sin(x)}{120x} = \lim_{x\to 0} \frac{\cos(x)}{120} = \frac{1}{120}.$$

It is not a coincidence that $r = \sin'''(0)/3!$ and that the limit is $\sin^{(5)}(0)/5!$.

EC3. (25 extra credit points) A four dimensional box has a cubical base and no top. Its four dimensional volume is supposed to be $8\,\text{ft}^4$. What dimensions minimize the three dimensional volume of material needed to make its base and sides?

Hint: A box in four dimensional space has 8 three dimensional "faces", each of which has the shape of a three dimensional box.

*Solution:* Since I can't draw a four dimensional box, there is no picture in this solution.

Let $x$ be the side length of the base, and let $y$ be the height. The base has volume $x^3$. There are 6 "sides", each with a volume of $x^2 y$. The four dimensional volume is $x^3 y$. So we are supposed to minimize the three dimensional volume $v = x^3 + 6x^2 y$ subject to the restriction $x^3 y = 8$.

The restriction is $y = 8x^{-3}$, so we are minimizing

$$v(x) = x^3 + 6x^2(8x^{-3}) = x^3 + 48x^{-1}.$$

The constraints are $x > 0$ and $y > 0$. The constraint $y > 0$ adds nothing new, so we are minimizing $v(x) = x^3 + 48x^{-1}$ on the interval $(0, \infty)$.

Differentiate: $v'(x) = 3x^2 - 48x^{-2}$. So we solve:

$$3x^2 - 48x^{-2} = 0$$
$$3x^4 - 48 = 0$$
$$x^4 = 16$$
$$x = 2 \qquad \text{or} \qquad x = -2.$$

We reject $x = -2$ because it is not in $(0, \infty)$. (I must see you do this! Otherwise, I don't know that you correctly solved the equation $v'(x) = 0$.) So 2 is the only critical number in the interval $(0, \infty)$.

Is it a maximum or minimum? The easiest test to use here is the second derivative test. One easily checks that $v''(x) = 6x + 96x^{-3}$, which is positive for all $x$ in the interval $(0, \infty)$. Therefore $v$ is concave up everywhere on the interval $(0, \infty)$, and any critical number must be an absolute minimum.

The problem asked for the dimensions giving the minimum three dimensional volume. The base is 2 feet by 2 feet by 2 feet. Since $y = 8x^{-3}$, the height is 1 foot. (Don't forget the units!)

For those who tried it instead of the second derivative test as done above, here is how the first derivative test works. We factor the derivative as $v'(x) = 3x^{-2}(x^4 - 16)$. All factors except $x^4 - 16$ are always positive. For $0 < x < 2$, the factor $x^4 - 16$ is negative. Thus $v'(x) < 0$, and $v$ is decreasing on the whole interval $(0, 2)$. For $x > 2$, the factor $x^4 - 16$ is positive. Thus $v'(x) > 0$, and $v$ is increasing on the whole interval $(2, \infty)$. Clearly, then, there is an absolute minimum at $x = 2$.

Other tests are also possible, but the details are not given here.