Math 251 Midterm 1 NAME: SOLUTIONS
F 20 Oct. 2017 Student ID: $\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$

Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Monday, this means complaints must be received by Tuesday 27 Oct.

1. (a) (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (14 points) If $f(x) = \dfrac{1}{8-x}$, compute the derivative $f'(2)$ *directly from the definition*. (You can check your answer using a differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient. To handle the expression that appears in the difference quotient, we subtract the fractions in the numerator and then cancel common factors in the numerator and denominator:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{8-(2+h)} - \frac{1}{8-2}}{h} = \lim_{h \to 0} \frac{\frac{1}{6-h} - \frac{1}{6}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{6-(6-h)}{6(6-h)}\right)}{h} = \lim_{h \to 0} \frac{\left(\frac{h}{6(6-h)}\right)}{h} = \lim_{h \to 0} \frac{1}{6(6-h)} = \frac{1}{36}. \end{aligned}$$

We can check using the differentiation formulas. It is easiest to use the chain rule: $f(x) = (8-x)^{-1}$, so $f'(x) = -(8-x)^{-2} \cdot (-1) = (8-x)^{-2}$, whence $f'(2) = \frac{1}{36}$. It can also be done using the quotient rule. (However, you get no credit if this is the only thing you do.)

2. (9 points/part) Differentiate the following functions:

(a) $g(t) = ae^t - \dfrac{7}{t^2} + \sqrt{t} + \pi^2$. ($a$ is a constant.)

Solution: We rewrite the second and third terms of the function to make it easy to differentiate:

$$g(t) = ae^t - 7t^{-2} + t^{1/2} + \pi^2.$$

Then

$$g'(t) = ae^t - 7 \cdot (-2)t^{-3} + \tfrac{1}{2}t^{1/2-1} = ae^t + 14t^{-3} + \tfrac{1}{2}t^{-1/2}.$$

The expression π^2 is a *constant*, so its derivative is zero. It is also possible to differentiate the second term using the quotient rule, but the solution given above is much faster.

(b) $h(x) = \sin(6x^2 - 11x)$.

Solution: Use the chain rule:

$$h'(x) = \cos(6x^2 - 11x) \cdot \frac{d}{dx}(6x^2 - 11x) = \cos(6x^2 - 11x) \cdot (6 \cdot 2x - 11) = (12x - 11)\cos(6x^2 - 11x).$$

3. (10 points) Find the equation of tangent line to the graph of $f(x) = x^2 - 2x$ at $x = -3$. You need not calculate the derivative directly from the definition.

Solution: We need the slope, which is $f'(-3)$, and a point on the tangent line, such as $(-3,\,(-3)^2-2(-3)) = (-3,\,15)$. Now $f'(x) = 2x-2$, so $f'(-3) = -8$. Therefore the equation is $y - 15 = -8(x-(-3))$, which can be rearranged to give $y = -8x - 9$.

Note that we want the slope at the *particular* value $x = -3$. Therefore we must substitute $x = -3$ in the formula for the derivative $f'(x)$ *before* using it as the slope of a line. The equation

$$\cancel{y - 15 = (2x-2)(x-(-3))}$$

is wrong—it is not even the equation of a line.

4. (9 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-x-6}$.

Solution: Both the numerator and denominator are continuous at 1, and the denominator is not zero there. Therefore the limit can be evaluated by simply substituting $x = 1$. That is,

$$\lim_{x\to 1} \frac{x-1}{x^2-x-6} = \frac{1-1}{1^2-1-6} = 0.$$

(b) $\displaystyle\lim_{x\to 10} \frac{x-10}{3\big(\sqrt{x}-\sqrt{10}\big)}$.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We rationalize the numerator, and then cancel common factors:

$$\begin{aligned}\lim_{x\to 10} \frac{x-10}{3\big(\sqrt{x}-\sqrt{10}\big)} &= \lim_{x\to 10} \frac{(x-10)\big(\sqrt{x}+\sqrt{10}\big)}{3\big(\sqrt{x}-\sqrt{10}\big)\big(\sqrt{x}+\sqrt{10}\big)} \\ &= \lim_{x\to 10} \frac{(x-10)\big(\sqrt{x}+\sqrt{10}\big)}{3(x-10)} = \lim_{x\to 10} \frac{\sqrt{x}+\sqrt{10}}{3} = \frac{2\sqrt{10}}{3}.\end{aligned}$$

Alternate solution: We start by factoring the numerator:

$$\lim_{x\to 10} \frac{x-10}{3\big(\sqrt{x}-\sqrt{10}\big)} = \lim_{x\to 10} \frac{\big(\sqrt{x}-\sqrt{10}\big)\big(\sqrt{x}+\sqrt{10}\big)}{3\big(\sqrt{x}-\sqrt{10}\big)} = \lim_{x\to 10} \frac{\sqrt{x}+\sqrt{10}}{3} = \frac{2\sqrt{10}}{3}.$$

(b) $\displaystyle\lim_{x\to \infty} \frac{x+109}{7x+1}$. (Be sure to show your work!)

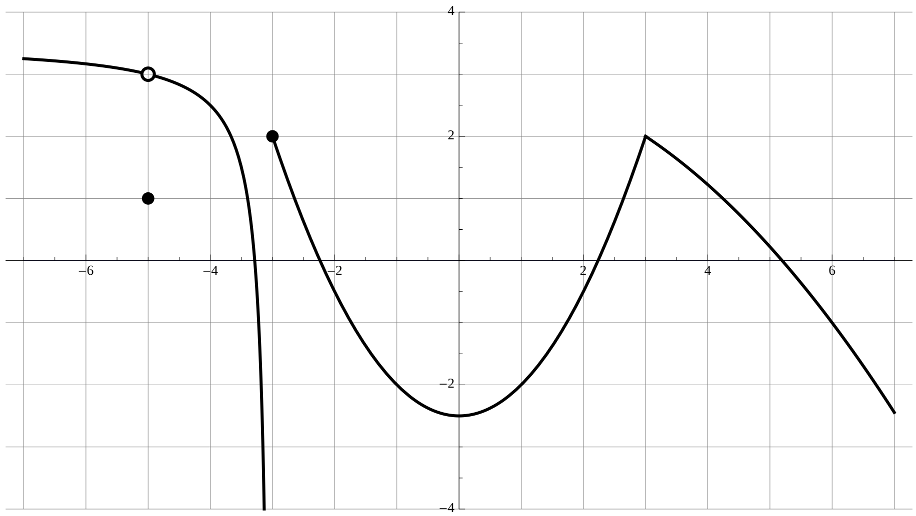
Solution: This has the indeterminate form "$\frac{\infty}{\infty}$", so work is needed. We factor out x from both the numerator and denominator, and then use the limit laws:

$$\lim_{x\to\infty} \frac{x+109}{7x+1} = \lim_{x\to\infty} \frac{x\left(1+\frac{109}{x}\right)}{x\left(7+\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1+\frac{109}{x}}{7+\frac{1}{x}} = \frac{1+\lim_{x\to\infty}\frac{109}{x}}{7+\lim_{x\to\infty}\frac{1}{x}} = \frac{1+0}{7+0} = \frac{1}{7}.$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{x\to\infty} \frac{x+109}{7x+1} = \lim_{x\to-\infty} \frac{\left(\frac{1}{x}\right)(x+109)}{\left(\frac{1}{x}\right)(7x+1)} = \lim_{x\to\infty} \frac{1+\frac{109}{x}}{7+\frac{1}{x}} = \frac{1+\lim_{x\to\infty}\frac{109}{x}}{7+\lim_{x\to\infty}\frac{1}{x}} = \frac{1+0}{7+0} = \frac{1}{7}.$$

5. For the function $y = k(x)$ graphed below, answer the following questions:



(a) (4 points.) Find $\lim_{x\to -5} k(x)$.

Solution: $\lim_{x\to -5} k(x) = 3$. (Note that it is different from $f(-5) = 1$.)

(b) (4 points.) Which of the following best describes $k'(4)$?
(1) $k'(4)$ does not exist.
(2) $k'(4)$ is close to 0.
(3) $k'(4)$ is positive and not close to 0.
(4) $k'(4)$ is negative and not close to 0.

Solution: $k'(4)$ is the slope of the tangent line to the graph of $y = k(x)$ at $x = 4$. You can tell from inspection that this slope is negative and not close to 0 (choice (4) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around $-8/9$. (In fact, it is clear that $k'(x) < 0$ for $3 < x < 7$.)

6. (4 points/part) A traffic reporter's helicopter is hovering over a freeway interchange. Its height above the ground varies. During the period from 8:00 am to 8:22 am, its height $y(t)$ above the ground, measured in meters, at time t, measured in minutes (min) after 8:00 am, is given by $y(t) = t^3 - 5t^2 + 110$.

(a) Is the helicopter falling or rising 2 minutes after 8:00 am? How fast?

Solution: Let $y(t)$ be the height, in feet, of the monster t seconds after it is thrown. Then $y(t) = 40t - 5t^2$ for t at most the time at which it hits the ground. Since $y(3) = (40)(3) - (5)(3^2) = 120 - 75 > 0$, after 3 seconds it hasn't hit the ground yet. Therefore the vertical velocity at time t is $y'(t) = 40 - 10t$, and the vertical velocity at time 3 is $y'(3) = 40 - (10)(3) = 10$. Therefore the monster is rising at 10 feet/second.
Note: You *must* include the units in this kind of problem.

(b) What is the average upwards velocity of the helicopter between 8:00 am and 8:02 am?

Solution: The average velocity is the how far it went divided by how long it took to go that far, which here is

$$\begin{aligned}\frac{y(2) - y(0)}{2 - 0} &= \frac{2^3 - (5)(2^2) + 110 - [0^3 - (5)(0^2) + 110]}{2} \\ &= \frac{8 - 20 + 110 - [0 - 0 + 110]}{2} = -6.\end{aligned}$$

So the average upwards velocity is $-16/9$ meters/min.
Note: You *must* include the units in this kind of problem.

It is not correct to average the velocities at the endpoints of the interval. That is, do not use

$$\cancel{\frac{y'(2)+y'(0)}{2}} = -4.$$

7. (9 points) If $xy = \cos(x+y) + \sin(6)$, find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

Solution: Differentiate both sides with respect to x, using the product rule on the left and the chain rule on the right:

$$1 \cdot y + x\frac{dy}{dx} = -\sin(x+y)\frac{d}{dx}(x+y) = -\sin(x+y)\left(1 + \frac{dy}{dx}\right).$$

(The derivative of $\sin(6)$ is immediately seen to be zero because $\sin(6)$ is a constant.) Now solve for $\dfrac{dy}{dx}$:

$$\begin{aligned} y + x\frac{dy}{dx} &= -\sin(x+y) - \sin(x+y)\frac{dy}{dx} \\ [x + \sin(x+y)]\frac{dy}{dx} &= -\sin(x+y) - y \\ \frac{dy}{dx} &= \frac{-\sin(x+y) - y}{x + \sin(x+y)} = -\frac{y + \sin(x+y)}{x + \sin(x+y)}. \end{aligned}$$

This fraction can't be further simplified.

For those who prefer the other notation, here it is written with y as an explicit function $y(x)$ of x. Start with

$$xy(x) = \cos(x + y(x)) + \sin(6).$$

Then differentiate with respect to x, just as before:

$$1 \cdot y(x) + xy'(x) = -\sin(x+y(x))\frac{d}{dx}(x+y(x)) = -\sin(x+y(x))\left(1 + y'(x)\right).$$

Now solve for $y'(x)$:

$$\begin{aligned} y(x) + xy'(x) &= -\sin(x+y(x)) - \sin(x+y(x))y'(x) \\ [x + \sin(x+y(x))]y'(x) &= -\sin(x+y(x)) - y(x) \\ y'(x) &= \frac{-\sin(x+y(x)) - y(x)}{x + \sin(x+y(x))} = -\frac{y(x) + \sin(x+y(x))}{x + \sin(x+y(x))}. \end{aligned}$$

As before, this fraction can't be further simplified.

EC1. (5 extra credit points) Let $f(x) = \cos(3x)$. Find the 1033th derivative $f^{(1033)}(x)$.

Solution: We have

$$\begin{aligned} f'(x) &= -3\sin(3x), \\ f''(x) &= -3^2\cos(3x), \\ f'''(x) &= 3^3\sin(3x), \end{aligned}$$

and

$$f^{(4)}(x) = 3^4\cos(3x) = 3^4 f(x).$$

Therefore

$$\begin{aligned} f^{(8)}(x) &= 3^4 f^{(4)}(x) = 3^8 f(x), \\ f^{(12)}(x) &= 3^8 f^{(4)}(x) = 3^{12} f(x), \end{aligned}$$

etc. In particular, since 1032 is divisible by 4, we get $f^{(1032)}(x) = 3^{1032}f(x) = 3^{1032}\cos(3x)$. Therefore

$$f^{(1033)}(x) = 3^{1032}f'(x) = 3^{1032}\cdot(-3\sin(3x)) = -3^{1033}\sin(3x).$$

EC2. (10 extra credit points) We will see later this quarter that if g is a differentiable function on an open interval (a,b), and if $g'(x) = 0$ for all x in (a,b), then g is constant. By considering the function $g(x) = \dfrac{f(x)}{e^x}$, prove that if f is a function on (a, b) such that $f'(x) = f(x)$ for all x, then there is a constant c such that $f(x) = ce^x$ for all x.

Solution: Let $g(x) = \dfrac{f(x)}{e^x}$. The denominator is nonzero on the entire interval (a, b). We therefore calculate, using the quotient rule and the relation $f'(x) = f(x)$:

$$g'(x) = \frac{f'(x)e^x - f(x)\frac{d}{dx}(e^x)}{(e^x)^2} = \frac{f(x)e^x - f(x)e^x}{(e^x)^2} = \frac{0}{(e^x)^2} = 0$$

for all x. Therefore g is constant on (a, b). Let c be the constant value. Then $\dfrac{f(x)}{e^x} = c$ for all x, so $f(x) = ce^x$ for all x.

EC3. (5 extra credit points/part) For each of the following parts, find a function f whose derivative is as given. Check your function to be sure its derivative really is what you think it is.

(a) $f'(x) = xe^{-x^2}$.

Solution: The function $f(x) = -\frac{1}{2}e^{-x^2}$ will work. Differentiate it (with the chain rule) and see!

(b) $f'(x) = x\sin(x)$.

Solution: The function $f(x) = -x\cos(x) + \sin(x)$ will work. Differentiate it to check. Note that the product rule says that the derivative of $-x\cos(x)$ is $-\cos(x) + x\sin(x)$. The derivative of $\sin(x)$ cancels out the first (unwanted) term.