Math 251 Midterm 2 NAME: SOLUTIONS

W 16 Nov. 2017 Student ID: $\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$

Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Friday, this means complaints must be received by Monday 19 Nov.

1. (25 points) A large spherical snowball was melting. (A child took it inside the house at 11:00 am, and his parents had not yet noticed.) At 11:07 am, its radius was 30 cm, and was decreasing at $\frac{1}{3}$ cm per minute. Was its surface area increasing or decreasing? At what rate? (Be sure to include the correct units in your answer.)

Note: There is no picture in this file.

Solution: Let $r(t)$ be the radius at time t, measured in centimeters and with time measured in minutes after 11:00 am, and let $S(t)$ be the surface area at time t, measured in square centimeters. Both the surface area and the radius vary with time, so must be treated as functions not constants.

The time we are interested in is $t = 7$. We are given $r(7) = 30$ and $r'(7) = -\frac{1}{3}$. The functions $S(t)$ and $r(t)$ are related by the equation

$$S(t) = 4\pi \left[r(t) \right]^2 .$$

(You are expected to know this formula.) Differentiate with respect to t:

$$S'(t) = 4\pi \cdot 2r(t)r'(t) = 8\pi r(t)r'(t).$$

(Don't forget the factor $r'(t)$! That will spoil the whole thing!) Evaluate this at $t = 7$, using $r(7) = 30$ and $r'(7) = -\frac{1}{3}$. This gives

$$S'(7) = 8\pi r(7)r'(7) = 8\pi \cdot 30 \cdot \left(-\frac{1}{3} \right) = -80\pi.$$

So the surface area is decreasing at the rate of 80π square centimeters per minute. (Don't forget the units!)

Here, for reference, is what the solution looks like in physicists' notation:

$$S = 4\pi r^2.$$

Differentiate with respect to t:

$$\frac{dS}{dt} = 4\pi \cdot 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

(Don't forget the factor $\dfrac{dr}{dt}$!) Now substitute $r = 30$ and $\dfrac{dr}{dt} = -\frac{1}{3}$, getting $\dfrac{dS}{dt} = 8\pi \cdot 30 \cdot \left(-\frac{1}{3}\right) = -80\pi$.

2. (10 points) Let $b(z) = \arctan(11) - z^2 \cos(\ln(z))$. Find $b'(z)$.

Solution: The derivative of $\arctan(11)$ is zero because $\arctan(11)$ is a constant. To differentiate the rest of it, we use the product rule, and then the chain rule on the second factor. This gives

$$\begin{aligned} b'(z) &= -\big(2z\cos(\ln(z)) + z^2\cos'(\ln(z))\ln'(z)\big) \\ &= -\left(2z\cos(\ln(z)) - z^2\sin(\ln(z))\left(\frac{1}{z}\right)\right) = -\big(2z\cos(\ln(z)) - z\sin(\ln(z))\big). \end{aligned}$$

The simplification in the last step is necessary. If you like, you rewrite this as

$$b'(z) = -2z\cos(\ln(z)) + z\sin(\ln(z)),$$

or factor as

$$b'(z) = z\big(\sin(\ln(z) - 2\cos(\ln(z))\big),$$

but these are not necessary.

3. The function $f(x)$ satisfies the following three properties:

$$f(3) = -2, \quad f'(3) = 0.50, \quad \text{and} \quad f''(3) = 1.02.$$

(a) (6 points) Use the linearization (tangent line approximation) to estimate the value of $f(2.96)$. (You might not need all the information given.)

Solution:

$$f(2.96) \approx f(3) + f'(3)(2.96 - 3) = -2 + (0.50)(-0.04) = -2.02.$$

(b) (4 points) Do you expect your answer in part (a) to be too big or too small? Why? (Drawing a picture might be useful.)

Solution: (There is no picture in the file.)

Since $f''(3) > 0$, the function is concave up at 3, so probably also near 3. This means its graph lies above the tangent line, so the answer in part (a) is probably too small.

4. (25 points) James Tutt Snodgrass III has finished Math 251 and gone home for the holidays to his mother's farm, only to be asked to solve the following problem:

His mother wants to fence off a rectangular enclosure for a llama. The west side of the enclosure will be a wall which is already present, but new fence must be built for the other three sides. She can afford 40 meters of fence. What is the largest possible total area she can enclose?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

(Show a full mathematical solution. A correct guess with no valid work will receive no credit, and a correct number supported only by heuristic reasoning will receive very little credit.)

Solution: Note: There is no picture in this file.

The arrangement of fences and the wall will be a rectangle with sides running north-south and east-west. We are supposed to maximize the area; call it A.

Let x be the length of the enclosure in the east-west direction, and let y be its length in the north-south direction, both measured in meters. The area is then $A = xy$. We are to maximize this expression. It has too many variables, and ignores the fact that x and y are not independent. There is no fence on the west side. (This is where the wall is.) The east side has length x meters, The south side has length y meters, and the north side also has length y meters. Adding these together give a total length of fence of $x + 2y$. Thus $x + 2y = 40$. It is easiest to solve for x, getting $x = 40 - 2y$, Substitute this for x in the formula $A = xy$, write it as a function of y, and multiply out for convenience:

$$A(y) = (40 - 2y)y = 40y - 2y^2.$$

The constraints are that x and y are both nonnegative. (Allowing the "degenerate" cases $x = 0$ and $y = 0$ will allow us to find the maximum over a closed bounded interval, which simplifies later steps.) The significance of the requirement $y \geq 0$ is clear, but we must express the requirement $x \geq 0$ in terms of y. Since $x = 40 - 2y$, it says that $2y \leq 40$, so $y \leq 20$. Our problem is therefore to maximize the function $A(y) = 40y - 2y^2$ for y in the closed bounded interval $[0, 20]$.

We search for critical numbers. Differentiate:

$$A'(y) = 40 - 4y.$$

Set the derivative equal to zero and solve:

$$0 = A'(y) = 40 - 4y$$
$$y = 10.$$

Since we are maximizing over a closed bounded interval, we need only compare the numbers $A(0)$, $A(10)$, and $A(10)$. These are

$$A(0) = 40 \cdot 0 - 2 \cdot 0^2 = 0, \quad A(20) = 40 \cdot 20 - 2 \cdot 20^2 = 800 - 800 = 0,$$

and

$$A(10) = 40 \cdot 10 - 2 \cdot 10^2 = 400 - 200 = 200.$$

Clearly the largest value is at $y = 10$. Therefore the north-south length should be 10 meters, and the east-west length should be $40 - 2 \cdot 10 = 20$ meters. It follows that the largest possible area is $A(10) = 200$ square meters. (Include the units!)

For reference, here are the other two approaches to test for a maximum. For both, we start at the point above where we found that $y = 10$ is the only number in our interval with $A'(y) = 0$. In both cases, one must still find the dimensions as above. Note that both of these also work over the *open* interval $(0, 20)$, and so could have been done without allowing the degenerate cases $x = 0$ and $y = 0$.

First derivative method: We know $A'(y) = 40 - 4y$. So $A'(y) > 0$ for $0 \leq y < 10$, and $A'(y) < 0$ for $y > 10$. This shows that $A(y)$ is increasing for $0 \leq y < 10$ and decreasing for $y > 10$. So $A(10)$ must be the largest value of $A(y)$ for y in $[0, 20]$.

Second derivative method: We know $A'(y) = 40 - 4y$. So $A''(y) = -4$. This is negative on the entire interval $(0, 20)$, so $A(y)$ is concave down there, and any number y with $A'(y) = 0$ must give a global maximum.

5. (30 points) Graph the function $f(x) = x^3 + 3x^2$ using the methods of calculus. In particular, determine exactly the x-intercept(s), y-intercept(s), asymptotes, intervals of increase and decrease, critical numbers, local maximums, local minimums, intervals of concavity, and inflection points. Be sure to label your axes, and include the scales. **Be sure to organize your work so that it is easy to follow, and explain what you are doing. A list of equations, with no explanation of how they relate to each other or to the problem, will receive little credit.**

Fill in the table (writing "none" if appropriate), and show full work and the graph elsewhere on the page.

Domain.	
x-intercepts.	
y-intercepts.	
Horizontal asymptotes.	
Vertical asymptotes.	
Critical numbers.	
Local minimums.	
Local maximums.	
Intervals of strict increase.	
Intervals of strict decrease.	
Intervals of upwards concavity.	
Intervals of downwards concavity.	
Inflection points.	

Solution: The domain clearly consists of all real numbers.

The y-intercept is $f(0) = 0$.

To find the x-intercepts, we solve $f(x) = 0$:

$$x^3 + 3x^2 = 0$$
$$x^2(x + 3) = 0$$

So the x-intercepts are at $x = 0$ and $x = -3$.

There are no vertical asymptotes, since f is defined and continuous on $(-\infty, \infty)$.

For horizontal asymptotes, we rewrite:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^2(x + 3) \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2(x + 3)$$

Since

$$\lim_{x \to \infty} x^2 = \infty, \quad \lim_{x \to -\infty} x^2 = \infty, \quad \lim_{x \to \infty} (x + 3) = \infty, \quad \text{and} \quad \lim_{x \to -\infty} (x + 3) = -\infty,$$

it follows that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^2(x + 3) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2(x + 3) = -\infty.$$

Therefore there are no horizontal asymptotes.

Next, we need the derivative:

$$f'(x) = 3x^2 + 6x.$$

To find the critical numbers, we solve $f'(x) = 0$:

$$3x^2 + 6x = 0$$

$$3x(x+2) = 0$$

$$x = 0 \quad \text{or} \quad x = -2.$$

So the critical numbers are $x = 0$ and $x = -2$.

On the interval $(-\infty, -2)$, the factor $3x$ in the expression $f'(x) = 3x(x+2)$ is negative and the factor $x - 2$ is also negative. So f' is positive on this interval. Therefore f is strictly increasing on this interval.

On the interval $(-2, 0)$, the factor $x+2$ above is positive and the factor x above is negative. So f' is negative on this interval. Therefore f is strictly decreasing on this interval.

On the interval $(0, \infty)$, the factor $3x$ is positive and the factor $x+2$ is also positive. So f' is positive on this interval. Therefore f is strictly increasing on this interval.

It now follows that f has a local maximum at $x = -2$ and a local minimum at $x = 0$. For use in graphing, we compute $f(0) = 0$ and $f(-2) = 4$.

Now, we need the second derivative:

$$f''(x) = 6x + 6.$$

This is zero exactly when $x = -1$.

On the interval $(-\infty, -1)$, that is, for $x < -1$, $f''(x)$ is clearly negative. Therefore f is concave down on this interval.

On the interval $(-1, \infty)$, that is, for $x > -1$, $f''(x)$ is clearly positive. Therefore f is concave up on this interval.

It follows that there is an inflection point at $x = -1$. For use in graphing, we compute $f(-1) = 2$.

Here is the filled in table.

Domain.	All real numbers
x-intercepts.	-3 and 0
y-intercepts.	0
Horizontal asymptotes.	none
Vertical asymptotes.	none
Critical numbers.	-2 and 0
Local minimums.	0
Local maximums.	-2
Intervals of strict increase.	$(-\infty, -2)$ and $(0, \infty)$
Intervals of strict decrease.	$(-2, 0)$
Intervals of upwards concavity.	$(-1, \infty)$
Intervals of downwards concavity.	$(-\infty, -1)$
Inflection points.	1

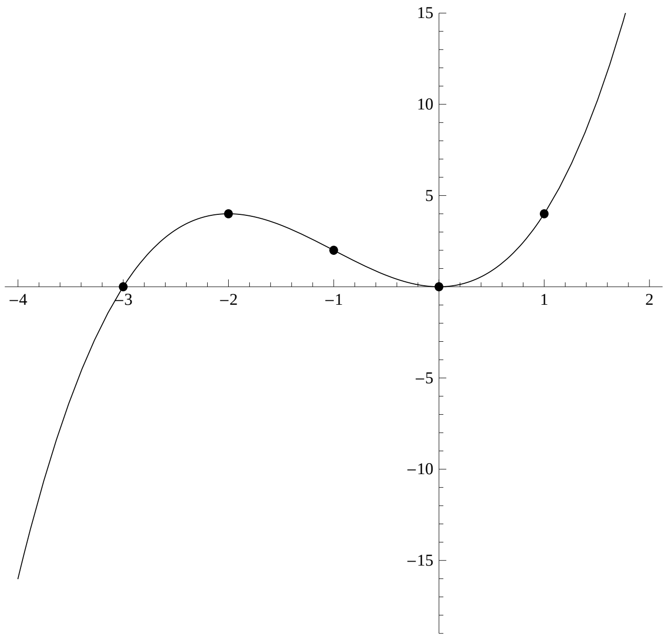
It isn't correct (despite what WeBWorK says) to say that

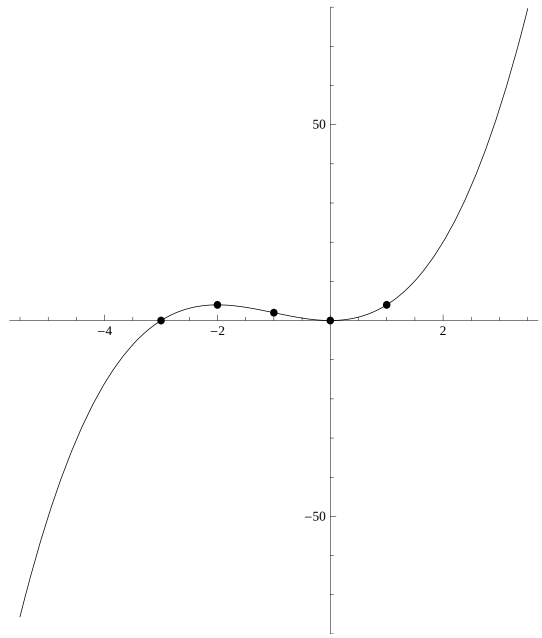
$$f \text{ is increasing on } (-\infty, -2) \cup (0, \infty),$$

since, as you can see from the graph, that isn't true. For example, $-2.1 < 0.1$ but $f(-2.1) > f(0.1)$.

Nosw we are ready to plot the graph. We should plot at least one point to the right of the larger critical number. I chose $x = 1$, giving $f(x) = 4$.

Here are two versions of the graph, showing as black dots the the point on the graph corresponding to the inflection point, the point on the graph corresponding to the local minimum, the point on the graph corresponding to the local maximum, the points on the graph corresponding to the x- and y-intercepts, and the point on the graph corresponding to $x = 1$. (Note that there is some repetition in this list, and that you were only asked to produce one graph.)





EC1. (20 extra credit points.) A 5 meter long ladder leans against a vertical wall in a room with a high ceiling and level floor. Because the floor is slippery, the foot of the ladder is sliding away from the wall. When the foot of the ladder is 3 meters from the wall, it is sliding away at $\frac{4}{3}$ meters per hour, and this motion is slowing down at $\frac{1}{3}$ meters/hour^2. At this time, is the motion of the top of the ladder speeding up or slowing down? How fast? (Be sure to include correct units.)

Solution: Note: There is no picture in this file.

Let $x(t)$ be the distance of the meter of the ladder from the base of the wall, in meters. Let $y(t)$ be the height above the floor of the top of the ladder, also in meters. Recall that the ladder is 5 meters long, and this length does not change. The missing picture should therefore show a right triangle, with horizontal side labelled $x(t)$, vertical side labelled $y(t)$, and hypotenuse (diagonal) labelled 5.

Let t_0 be the time at which we are interested. We are given the following:

$$x(t_0) = 3, \quad x'(t_0) = \frac{4}{3}, \quad \text{and} \quad x''(t_0) = -\frac{1}{3}.$$

(The derivative $x'(t_0)$ is positive because the distance from the foot of the ladder to the wall is increasing. The second derivative $x''(t_0)$ is negative because the rate at which the foot of the ladder is moving is decreasing.)

Our quantities are related by the equation

$$[x(t)]^2 + [y(t)]^2 = 5^2, \tag{1}$$

since we have a right triangle. Differentiating this equation with respect to t gives

$$2x(t)x'(t) + 2y(t)y'(t) = 0.$$

(Don't forget to use the chain rule!) Dividing through by 2 gives

$$x(t)x'(t) + y(t)y'(t) = 0. \tag{2}$$

Differentiating with respect to t again gives

$$[x'(t)]^2 + x(t)x''(t) + [y'(t)]^2 + y(t)y''(t) = 0. \tag{3}$$

Substituting t_0 for t in (1) and using $x(t_0) = 3$ gives $y(t_0) = 4$. (We reject the solution $y(t_0) = -4$, because $y(t)$ is positive.) Substituting t_0 for t in (2), and using $x(t_0) = 3$ and $y(t_0) = 4$, gives

$$3 \cdot \left(\frac{4}{3}\right) + 4y'(t_0) = 0.$$

Therefore $y'(t_0) = -1$. Substituting t_0 for t in (3), and using

$$x(t_0) = 3, \quad y(t_0) = 4, \quad x'(t_0) = \frac{4}{3}, \quad y'(t_0) = -1, \quad \text{and} \quad x''(t_0) = -\frac{1}{3},$$

gives

$$\left(\frac{4}{3}\right)^2 + 3\left(-\frac{1}{3}\right) + (-1)^2 + 4y''(t_0) = 0.$$

This simplifies to

$$\frac{16}{9} - 1 + 1 + 4y''(t_0) = 0,$$

so $y''(t_0) = -\frac{4}{9}$. Therefore the rate at which the top of the ladder is sliding down the wall is decreasing at $\frac{4}{9}$ meters/hour^2. (The units are required.)

EC2. (20 extra credit points.) Either draw the graph of a function satisfying the given conditions, or prove that no such function exists: $f''(x)$ exists for all $x > 1$, f is concave up on $[1, \infty)$, $\lim_{x\to\infty} f(x) = 3$, and $f(x) < 3$ for all $x > 1$. (Vague explanations and graphs not satisfying all the conditions will get no credit.)

Solution: No such function is possible.

By hypothesis, $f(2) < 3$. Since $\lim_{x\to\infty} f(x) = 3$, there is some $x_0 > 2$ such that $f(x_0) > f(2)$. The Mean Value Theorem now provides c in the interval $(2, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(2)}{x_0 - 2}.$$

In particular, $f'(c) > 0$.

Set $\alpha = \frac{1}{2}f'(c) > 0$. Define a function g by $g(x) = f(x) - \alpha x$. Since f is concave up on $(1, \infty)$, its derivative f' must at least be nondecreasing on $(1, \infty)$. Therefore $f'(x) \geq f'(c) > \alpha$ for all $x > c$. It follows that $g'(x) > 0$ for all $x > c$. Thus g is strictly increasing. In particular, for all $x > c$ we have $g(x) > g(c)$. Substituting the definition of g, we see that for all $x > c$ we have $f(x) - \alpha x > f(c) - \alpha c$. Rearranging, we get $f(x) > f(c) - \alpha c + \alpha x$. Since $\alpha > 0$, we have $\lim_{x\to\infty} \alpha x = \infty$. Since $f(c) - \alpha c$ is a constant, it follows that $\lim_{x\to\infty} f(x) = \infty$. But $\lim_{x\to\infty} f(x)$ was supposed to be 3.