WORKSHEET SOLUTIONS: LIMITS 1

Names and student IDs: Solutions [πππ-ππ-ππππ]

1. You want to find $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 2}$. Set $f(x) = \frac{x^2 - 3x + 2}{x - 2}$ for $x \neq 2$.

Step 1: Does anything go wrong if you try to substitute $x = 1$?

Solution. No, $f(1)$ is defined and

$$f(1) = \frac{1^2 - 3 \cdot 1 + 2}{1 - 2} = \frac{0}{-1} = 0.$$

□

Step 2: Your answer above should have been "no". So what do you think the limit should be?

Solution. You expect

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 2} = \frac{1^2 - 3 \cdot 1 + 2}{1 - 2} = \frac{0}{-1} = 0.$$

(As we will see, this is in fact correct.) □

2. You want to find $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$.

Step 1: Does anything go wrong if you try to substitute $x = 2$?

Solution. Yes, the numerator is $2^2 - 3 \cdot 2 + 2 = 0$ and the denominator is $2 - 2 = 0$. Therefore the limit has the indeterminate form "$\frac{0}{0}$", so more work is needed. □

Don't ever write either of

$$\cancel{\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \frac{0}{0}} \qquad \text{or} \qquad \cancel{\frac{x^2 - 3x + 2}{x - 2} = \frac{0}{0}}.$$

The expression '$\frac{0}{0}$' can't ever appear in a mathematical equation, because it doesn't have a meaning.

Step 2: Your answer above should have been "yes". So what do you do? Hint: Factor the numerator.

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Solution.

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\lim_{x\to 2}\frac{(x-1)(x-2)}{x-2}=\lim_{x\to 2}(x-1).$$

Now there is no difficulty with putting $x=2$. So you expect should be $2-1=1$. (As we will see, this is in fact correct.) $\square$

3. You want to find $\lim_{x\to 2}\dfrac{x^2-3x+2}{x^2-4}$.

Step 1: Does anything go wrong if you try to substitute $x=2$?

Solution. Yes, the numerator is $2^2-3\cdot 2+2=0$ and the denominator is $2^2-4=0$. Therefore the limit has the $\square$

indeterminate form "$\frac{0}{0}$", so more work is needed.

Step 2: Your answer above should have been "yes". So what is the first algebraic step you do?

Solution. Factor the numerator an denominator and cancel:

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\lim_{x\to 2}\frac{(x-1)(x-2)}{x^2-4}=\lim_{x\to 2}\frac{x-1}{x+2}.$$

(**Now** there is no difficulty with putting $x=2$.) $\square$

4. You want to find $\lim_{x\to 3}\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$.

Does anything go wrong if you try to substitute $x=3$?

Solution. Yes, the numerator is $\sqrt{3}-\sqrt{3}=0$ and the denominator is $3-3=0$. Therefore the limit has the indeterminate form "$\frac{0}{0}$", so more work is needed. $\square$

Your answer above should have been "yes". It isn't obvious how to factor, so let's try to estimate the limit numerically. Use a calculator to approimate the following:

$$f(2)\approx 0.3178 \qquad f(4)\approx 0.2679$$
$$f(2.9)\approx 0.2911 \qquad f(3.1)\approx 0.2863$$
$$f(2.99)\approx 0.2889 \qquad f(3.01)\approx 0.2884$$

What is your guess for the limit?

Solution. It looks like

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} \approx 0.2886.$$

□

Let's try to find the exact value. Rationalize the numerator: multiply the numerator and denominator by $\sqrt{x}-\sqrt{3}$. Multiply out in the numerator but **not** in the denominator.

Solution.

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3} \frac{\left(\sqrt{x}-\sqrt{3}\right)\left(\sqrt{x}+\sqrt{3}\right)}{(x-3)\left(\sqrt{x}+\sqrt{3}\right)} = \lim_{x\to 3} \frac{x-3}{(x-3)\left(\sqrt{x}+\sqrt{3}\right)} = \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

□

Alternate solution. This solution depends on factoring the denominator instead of the hint. It doesn't immediately look like it, but the denominator is a difference of squares:

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{\left(\sqrt{x}-\sqrt{3}\right)\left(\sqrt{x}+\sqrt{3}\right)} = \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

□

Suppose you **do** multiply out in the denominator. What do you get, and what do you do next?

Solution. You get

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3} \frac{\left(\sqrt{x}-\sqrt{3}\right)\left(\sqrt{x}+\sqrt{3}\right)}{(x-3)\left(\sqrt{x}+\sqrt{3}\right)} = \lim_{x\to 3} \frac{x-3}{x^{3/2}+\sqrt{3}x-3\sqrt{x}-3\sqrt{3}}.$$

This still has the indeterminate form "$\frac{0}{0}$", and it isn't obvious what to do next. In fact, you need to factor the denominator, exactly undoing part of the previous step. □