WORKSHEET SOLUTIONS: DERIVATIVES FROM THE DEFINITION

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Recall: if f is a function defined on an open interval containing a, then the derivative of f at a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

An alternate formulation is: the derivative of f at a is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

1. Let $k(x) = x^2$ for all real numbers x. Let a be an arbitrary real number. Find $k'(a)$ directly from the definition. You should get $k'(a) = 2a$.

Solution. We find the limit of the difference quotient. Direct substitution gives the expression "$\frac{0}{0}$". Therefore we use the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} k'(a) &= \lim_{h\to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{(a^2 + 2ah + h^2) - a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0} (2a + h) = 2a. \end{aligned}$$

□

2. If $k(x) = x^2$ for all real numbers x, then for all real numbers x, we have $k'(x) = \underline{\ 2x\ }$.

3. Expand the expression $(x+h)^3$.

Solution.

$$\begin{aligned} (x+h)^3 &= (x+h)(x+h)^2 = (x+h)(x^2 + 2xh + h^2) \\ &= (x^3 + 2x^2h + xh^2) + (x^2h + 2xh^2 + h^3) = x^3 + 3x^2h + 3xh^2 + h^3. \end{aligned}$$

□

4. Let $l(x) = x^3$ for all real numbers x. Let a be an arbitrary real number. Find $l'(x)$ directly from the definition. You should get $l'(x) = 3x^2$.

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Solution. We find the limit of the difference quotient. Direct substitution gives the expression "$\frac{0}{0}$". Therefore we use the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$l'(x) = \lim_{h\to 0} \frac{l(x+h) - l(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

□

5. For a nonnegative integer n, let $P_n(x)$ be the function $P_n(x) = x^n$ for all real numbers x. (Take $P_0(0) = 1$.)

You have seen $P_0'(x)$, $P_1'(x)$, $P_2'(x)$, and $P_3'(x)$. What do you think $P_4'(x)$ is? What do you think $P_n'(x)$ is?

Solution. The obvious guesses are $P_4'(x) = 4x^3$ and $P_n'(x) = nx^{n-1}$. These are in fact correct, even when n isn't an integer. □