MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 2.

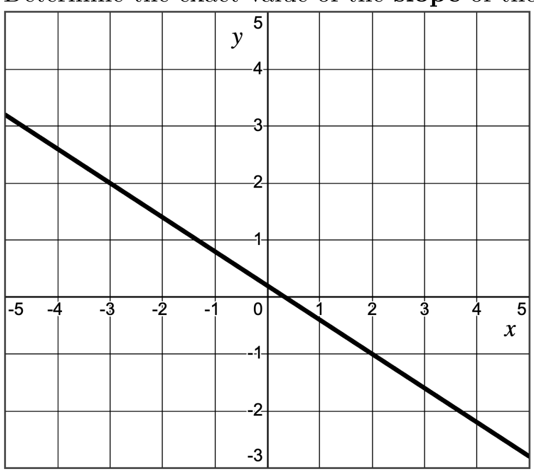
This homework sheet is due in class on Wednesday 15 January 2025 (week 2), in class. Write answers on a separate piece of paper, well organized and well labelled, with each solution starting on the left margin of the page.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

10 points per problem, total 50 points.

1. A version of Midterm Zero administered to a class of fire breathing monsters on the planet Yuggxth contained the following problem:

Determine the exact value of the **slope** of the line in the graph below.



Some of the papers turned in were too badly burned to be read, but among those that could be read, the answers listed below were found. Keeping in mind that the scales on the two axes are the same, four of these answers are obviously wrong, that is, can be seen to be wrong without calculating anything. For each of those four, explain why that answer is obviously wrong. Also, determine whether any of them is correct, stating, for each, "correct" or "wrong" as appropriate.

(a) $\frac{3}{5}$.

Solution: This answer is obviously wrong, because the slope of the line shown is obviously negative.

(b) -17.

Solution: This answer is obviously wrong, because a line of slope -17 is much steeper. Since the scales on the two axes are the same, the slope of the line shown is very roughly -1.

(c) $-\frac{1}{20}$.

Solution: This answer is obviously wrong, because a line of slope $-\frac{1}{20}$ is much flatter. Since the scales on the two axes are the same, the slope of the line shown is very roughly -1.

(d) $-\frac{3}{5}$.

Date: 15 January 2025 .

Solution: This answer is correct. You can tell by reading the graph that the points $(x_1, y_1) = (-3, 2)$ and $(x_2, y_2) = (2, -1)$ are on the line. Therefore the slope is

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{2 - (-1)}{-3 - 2} = \frac{3}{-5} = -\frac{3}{5}.$$

Another approach is to simply observe from the graph that the line goes down three units for each five units to the right.

(e) $y = -\frac{3}{5}x + \frac{1}{5}$.

Solution: This answer is obviously wrong. The question asks for the slope, which is a real number, and "$y = -\frac{3}{5}x + \frac{1}{5}$" is an equation, not a real number.

(f) $-\frac{4}{5}$.

Solution: This answer is wrong, but not obviously wrong. You ned to actually look carefully at the graph (see part (d) above) to see that the slope is actually $-\frac{3}{5}$.

2. Find the derivative of the function $g(t) = 4t^3 - at^2 + \pi^2$. ($a$ is a *constant*.)

Solution:

$$g'(t) = 4 \cdot 3t^2 - a \cdot 2t = 12t^2 + 2at.$$

The expression π^2 is a *constant*, so its derivative is zero.

3. This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\lim\limits_{x \to 3} \dfrac{x\cos(x) + x - 3\cos(x) - 3}{x\cos(x) + 2x - 3\cos(x) - 6}$. The method is to factor the numerator and denominator, and cancel the common factor. The factors of the numerator are $x - 3$ and $\cos(x) + 1$, and the factors of the denominator are $x - 3$ and $\cos(x) + 2$.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. Start at $\lim\limits_{x \to 3} \dfrac{x\cos(x) + x - 3\cos(x) - 3}{x\cos(x) + 2x - 3\cos(x) - 6}$. Show at least the following steps:

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x = 3$.
(3) After substituting $x = 3$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified. (Don't use a decimal approximation to $\cos(3)$, and don't give a decimal approximation to the final answer.)

Solution.

$$\begin{aligned}\lim_{x \to 3} \frac{x\cos(x) + x - 3\cos(x) - 3}{x\cos(x) + 2x - 3\cos(x) - 6} &= \lim_{x \to 3} \frac{(x-3)(\cos(x)+1)}{(x-3)(\cos(x)+2)} \\ &= \lim_{x \to 3} \frac{\cos(x)+1}{\cos(x)+2} = \frac{\cos(3)+1}{\cos(3)+2}.\end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq 3$, we have

$$\frac{x\cos(x) + x - 3\cos(x) - 3}{x\cos(x) + 2x - 3\cos(x) - 6} = \frac{(x-3)(\cos(x)+1)}{(x-3)(\cos(x)+2)} = \frac{\cos(x)+1}{\cos(x)+2}.$$

Therefore

$$\lim_{x \to 3} \frac{x\cos(x) + x - 3\cos(x) - 3}{x\cos(x) + 2x - 3\cos(x) - 6} = \lim_{x \to 3} \frac{\cos(x)+1}{\cos(x)+2} = \frac{\cos(3)+1}{\cos(3)+2}.$$

This completes the solution. □

Comments. The expression $\dfrac{\cos(3)+1}{\cos(3)+2}$ can't be simplified. In line with the instructions above, decimal approximations will be automatically marked wrong.

In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

The symbol "$\lim_{x\to 3}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\cancel{\lim_{x\to 3} \frac{(x-3)(\cos(x)+1)}{(x-3)(\cos(x)+2)} = \lim_{x\to 3} \frac{\cos(3)+1}{\cos(3)+2}}$$

is a mathematically true statement, but does not correctly show the intended step.

Every parenthesis shown is essential, except for the possibility of writing "$\cos x$" and "$\cos 3$" in place of "$\cos(x)$" and "$\cos(3)$", an unfortunately standard bad habit. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.

4. This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\lim_{x\to -1} \dfrac{x^7+x^6+5x+5}{x+1}$. The method is to factor the numerator and cancel one of the factors. The factors of the numerator are $x+1$ and x^6+5.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. Start at $\lim_{x\to -1} \dfrac{x^7+x^6+5x+5}{x+1}$. Show at least the following steps:

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x=-1$.
(3) After substituting $x=-1$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified.

Solution.

$$\begin{aligned} \lim_{x\to -1} \frac{x^7+x^6+5x+5}{x+1} &= \lim_{x\to -1} \frac{(x+1)(x^6+5)}{x+1} \\ &= \lim_{x\to -1} (x^6+5) = (-1)^6+5 = 1+5 = 6. \end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq -1$, we have

$$\frac{x^7+x^6+5x+5}{x+1} = \frac{(x+1)(x^6+5)}{x+1} = x^6+5.$$

Therefore

$$\lim_{x\to -1} \frac{x^7+x^6+5x+5}{x+1} = \lim_{x\to -1} (x^6+5) = (-1)^6+5 = 1+5 = 6.$$

This completes the solution. □

Comments. In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

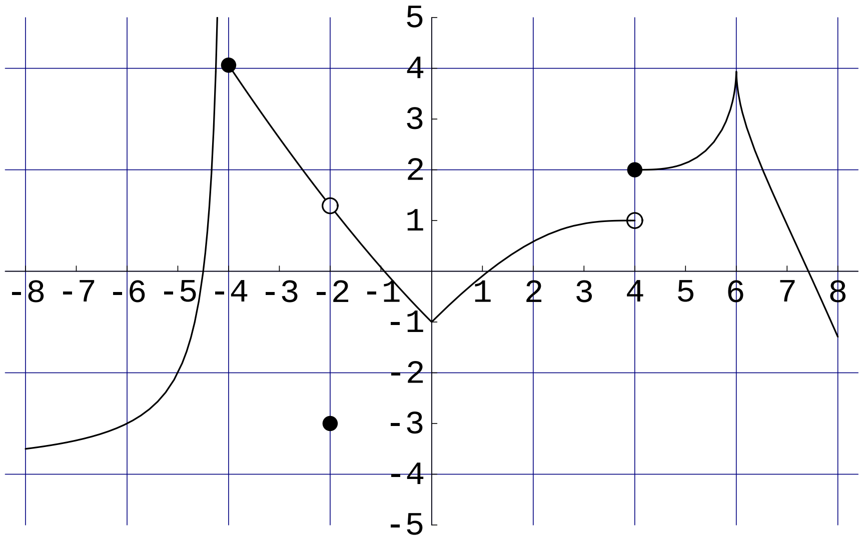
The symbol "$\lim_{x\to -1}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\cancel{\lim_{x\to -1} (x^6+5) = \lim_{x\to -1} [(-1)^6+5]}$$

is a mathematically true statement, but does not correctly show the intended step.

Every parenthesis shown is essential. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.

5. For the function $y = q(x)$ graphed below, answer the following questions.



(a) List all numbers a in $(-8, 8)$ such that q is not continuous at a. Give reasons.

Solution: The answer is $a = -4$, $a = -2$, and $a = 4$. The function q is not continuous at -4 and at 4, because $\lim_{x\to-4} q(x)$ and $\lim_{x\to4} q(x)$ do not exist. (We have $\lim_{x\to-4^-} q(x) = \infty$, and at 4 the two one sided limits exist but are not equal.) The function q is not continuous at -2 because, although $\lim_{x\to-2} q(x)$ exists, it is not equal to $q(-2)$. Note that q *is* continuous at 0 and at 6 (although q is not differentiable at those numbers).

(b) Find the largest interval containing -3 on which q is continuous.

Solution: The largest interval containing -3 on which q is continuous is $(-4, -2)$. The function q is not continuous at either -4 or -2, because $\lim_{x\to-4} q(x)$ does not exist and $\lim_{x\to-2} q(x) \neq q(-2)$.