WORKSHEET SOLUTIONS: PRODUCT AND QUOTIENT RULES

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall the most recent differentiation rules we have seen:

(1) If f and g are differentiable, and $j(x) = f(x)g(x)$ for all x (in a suitable open interval), then $j'(x) = f'(x)g(x) + f(x)g'(x)$.

(2) Quotient rule: If f and g are differentiable, $g(x)$ is never zero (on a suitable open interval) and

$$j(x) = \frac{f(x)}{g(x)}$$

for all x (in a suitable open interval), then

$$j'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

(3) The functions sin and cos are differentiable everywhere, and (using **radians!**)

$$\sin'(x) = \cos(x) \qquad \text{and} \qquad \cos'(x) = -\sin(x)$$

for all x.

Now differentiate the following functions, or else tell me that no differentiation rule you have seen so far applies:

To differentiate $s(x) = x^3 \sin(x)$, use the product rule. Write $s(x) = f(x)g(x)$ with

$$f(x) = x^3 \qquad \text{and} \qquad g(x) = \sin(x).$$

Thus:

$$s'(x) = f'(x)g(x) + f(x)g'(x) = \frac{d}{dx}\left(x^3\right)\sin(x) + x^3\sin'(x) = 3x^2\sin(x) + x^3\cos(x).$$

To differentiate $s(x) = \dfrac{\sin(x)}{x^2+1}$, use the quotient rule. Write $s(x) = \dfrac{f(x)}{g(x)}$ with

$$f(x) = \sin(x) \qquad \text{and} \qquad g(x) = x^2 + 1.$$

Thus:

$$s'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{\sin'(x)(x^2+1) - \sin(x)\dfrac{d}{dx}\left(x^2+1\right)}{(x^2+1)^2} = \frac{(x^2+1)\cos(x) - 2x\sin(x)}{(x^2+1)^2}.$$

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If $w(t) = (3t^2 + t)\cos(t) + t^6$ then use the product rule on the first term, with $f(t) = 3t^2 + t$ and $g(t) = \cos(t)$, so that $w(t) = f(t)g(t)$, to get

$$w'(t) = \frac{d}{dt}\big(3t^2 + t\big)\cos(t) + (3t^2 + t)\cos'(t) + \frac{d}{dt}\big(t^6\big) = (6t + 1)\cos(t) - (3t^2 + t)\sin(t) + 6t^5.$$

$$\frac{d}{dx}\big((x^2 + 3x)(11x^7 - 102x^3)\big) = (2x + 6)(11x^7 - 102x^3) + (x^2 + 3x)(77x^6 - 306x^2).$$

If $q(x) = \dfrac{x^3\sin(x)}{x^2 + 1}$, use both the product and quotient rules. The derivative $\dfrac{d}{dx}\big(x^3\sin(x)\big)$ was computed above, using the product rule: it is $\dfrac{d}{dx}\big(x^3\sin(x)\big) = 3x^2\sin(x) + x^3\cos(x)$. So

$$\begin{aligned} q'(x) &= \frac{\frac{d}{dx}\big(x^3\sin(x)\big)(x^2+1) - x^3\sin(x)\frac{d}{dx}\big(x^2+1\big)}{(x^2+1)^2} \\ &= \frac{\big[3x^2\sin(x) + x^3\cos(x)\big](x^2+1) - x^3\sin(x)\cdot 2x}{(x^2+1)^2} \\ &= \frac{\big[3x^2\sin(x) + x^3\cos(x)\big](x^2+1) - 2x^4\sin(x)}{(x^2+1)^2} \\ &= \frac{(x^4 + 3x^2)\sin(x) + (x^5 + x^3)\cos(x)}{(x^2+1)^2}. \end{aligned}$$

No differentiation rule we have seen so far tells us anything about $\dfrac{d}{dx}\big(7^x\big)$. (This function is **not** a general power: the derivative is **not** ~~$x \cdot 7^{x-1}$~~.)