WORKSHEET SOLUTIONS: CHAIN RULE

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Chain rule: If g is differentiable at x and f is differentiable at $g(x)$, and if

$$h(x) = f(g(x))$$

for all x (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Example for the chain rule: If $h(x) = \sin(x^3)$ then write $h(x) = f(g(x))$ with

$$f(x) = \sin(x) \qquad \text{and} \qquad g(x) = x^3.$$

Thus

$$h'(x) = f'(g(x)) \cdot g'(x) = \sin'(x^3) \cdot \frac{d}{dx}\big(x^3\big) = \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

The last step is conventional. Caution: "$\sin'(x^3)$" means you take the derivative of the sine function and evaluate it at x^3. It does *not* mean $\cancel{\frac{d}{dx}\big(\sin(x^3)\big)}$.

Now differentiate the following functions, or else tell me that no differentiation rule you have seen so far applies.

Let $w(x) = \cos(x^4)$. First, if we want to usefully write $w(x) = f(g(x))$, then

$$f(x) = \cos(x) \qquad \text{and} \qquad g(x) = x^4.$$

If $w(x) = \cos(x^4)$ then

$$w'(x) = \cos'(x^4) \cdot \frac{d}{dx}\big(x^4\big) = -\sin(x^4) \cdot 4x^3 = -4x^3 \sin(x^4).$$

Let $p(x) = (x^{11} + 3x + 1)^{109}$. First, if we want to usefully write $p(x) = f(g(x))$, then

$$f(x) = x^{109} \qquad \text{and} \qquad g(x) = x^{11} + 3x + 1.$$

The chain and power rules thus give

$$\begin{aligned} \frac{d}{dx}\big((x^{11} + 3x + 1)^{109}\big) &= 109(x^{11} + 3x + 1)^{108} \cdot \frac{d}{dx}\big(x^{11} + 3x + 1\big) \\ &= 109(x^{11} + 3x + 1)^{108}(11x^{10} + 3). \end{aligned}$$

(It is wrong if you leave out the parentheses in any of the steps.)

If $r(x) = \big(\sin(x) + 29\big)^{1/3}$, then

$$r'(x) = \frac{1}{3}\big(\sin(x) + 29\big)^{-2/3} \cdot \frac{d}{dx}\big(\sin(x) + 29\big) = \frac{1}{3}\big(\sin(x) + 29\big)^{-2/3} \cos(x).$$

If f is a function such that $f'(s) = e^{s^2}$ for all real s, find $\frac{d}{dx}\big(f(\tan(x))\big)$ and $\frac{d}{dx}\big(\tan(f(x))\big)$. (Recall that $\tan'(x) = \sec^2(x)$.)

For the first,

$$\frac{d}{dx}\big(f(\tan(x))\big) = f'(\tan(x)) \tan'(x) = e^{\tan^2(x)} \sec^2(x).$$

For the second,

$$\frac{d}{dx}\big(\tan(f(x))\big) = \tan'(f(x)) f'(x) = \sec^2(f(x)) e^{x^2}.$$

You can't do anything about the fact that f still appears in the answer.

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$$\frac{d}{dx}\big(x^{\sin(x)}\big) =$$

This requires a trick we haven't done yet: rewrite $x^{\sin(x)} = \exp(\ln(x)\sin(x))$. It also needs $\frac{d}{dx}\big(e^x\big) = e^x$ and $\ln'(x) = \frac{1}{x}$. Use the chain rule, and use the product rule on the insides:

$$\begin{aligned}\frac{d}{dx}\big(x^{\sin(x)}\big) &= \frac{d}{dx}\big(e^{\ln(x)\sin(x)}\big) = e^{\ln(x)\sin(x)}\frac{d}{dx}\big(\ln(x)\sin(x)\big) \\ &= e^{\ln(x)\sin(x)}\big(\ln'(x)\sin(x) + \ln(x)\sin'(x)\big) = e^{\ln(x)\sin(x)}\left(\frac{\sin(x)}{x} + \ln(x)\cos(x)\right).\end{aligned}$$