MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4

This homework sheet is due in class on Tuesday 28 January 2025 (week 3), in class. Write answers on a separate piece of 8.5 by 11 inch paper, well organized and well labelled, with each solution starting on the left margin of the page.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

Point values as indicated, total 50 points.

1. (10 points.) Find the derivative of the function $R(t) = 4at^3 - t^2\cos(t) - \pi^2$. ($a$ is a *constant*.)

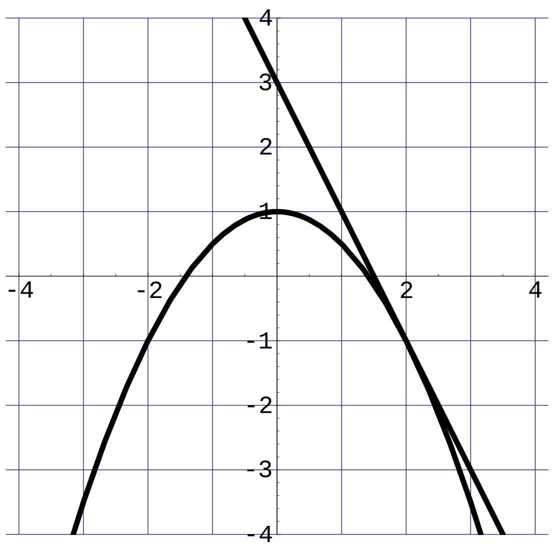
Solution: The product rule says

$$\frac{d}{dt}\big(t^2\cos(t)\big) = 2t\cos(t) + t^2(-\sin(t)) = 2t\cos(t) - t^2\sin(t).$$

Also, $\frac{d}{dt}\big(\pi^2\big) = 0$ because π^2 is a constant. Using the power and multiplication by a constant rules on the first term, this gives

$$R'(t) = 12at^3 - [2t\cos(t) - t^2\sin(t)] - 0 = 12at^3 - 2t\cos(t) + t^2\sin(t).$$

2. (10 points.) The picture below shows the graph of a function $y = f(x)$ and the tangent line to the graph at $x = 2$.



Let g be the function $g(x) = \dfrac{f(x)}{x^2+7}$. Find $g'(2)$.

Solution: FOR WRONG PROBLEM!
By the quotient rule, we have

$$g'(x) = \frac{f'(x)(x^2+7) - f(x)\frac{d}{dx}\big(x^2+7\big)}{(x^2+7)^2} = \frac{(x^2+7)f'(x) - 2xf(x)}{(x^2+7)^2}.$$

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Therefore

$$g'(2) = \frac{(2^2+7)f'(2) - 2\cdot 2f(2)}{(2^2+7)^2} = \frac{11f'(2) - 4f(2)}{11^2}.$$

We can read off $f(2) = -1$ from the graph.

We now need $f'(2)$. Examining the graph, we see that the tangent line goes through the points $(2, -1)$ and $(0, 3)$. Therefore its slope is $f'(2) = \frac{3-(-1)}{0-2} = -2$.

Now substitute:

$$g'(2) = \frac{11f'(2) - 4f(2)}{11^2} = \frac{11(-2) - 4(-1)}{121} = -\frac{18}{121}.$$

For reference, the function in the graph is $f(x) = 1 - \frac{1}{2}x^2$.

3. (10 points.) Let f and g be functions which are differentiable at -2 and which satisfy

$$f(-2) = -5, \qquad f'(-2) = -3, \qquad g(-2) = 4, \qquad \text{and} \qquad g'(-2) = 2.$$

Let $w(x) = x - f(x)g(x)$ for all x. Find $w'(-2)$.

Solution: Using the product rule on the second part, we get:

$$w'(x) = 1 - [f'(x)g(x) + f(x)g'(x)] = 1 - f'(x)g(x) - f(x)g'(x).$$

This gives

$$w'(-2) = 1 - f'(-2)g(-2) - f(-2)g'(-2) = 1 - (-5)(4) - (-3)(2) = 27.$$

4. (20 points.) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

Be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: There is no picture in this file.

We are supposed to maximize the volume of the cylinder. Let's call it V.

Let's decide that the cylinder is supposed to be inscribed vertically, that is, with the flat (circular) ends at the top and bottom. Further let x be the distance from the center of the sphere to the top of the cylinder. This is *half* the height of the cylinder. Let y be the radius of the cylinder. Then V is the height $2x$ multiplied by the area πy^2 of the base, that is, $V = 2x \cdot \pi y^2 = 2\pi x y^2$.

We must relate x and y. The Pythagorean Theorem gives $x^2 + y^2 = r^2$. Since y^2 already appears in the formula above, but y by itself does not, the easiest way to proceed is:

$$y^2 = r^2 - x^2$$

$$V = 2\pi x y^2 = 2\pi x(r^2 - x^2)$$

Rewriting it for easy differentiation:

$$V(x) = 2\pi r^2 x - 2\pi x^3.$$

Now we consider the allowed values of x. Clearly $x \geq 0$. Also $x \leq r$, because otherwise the cylinder would stick outside the sphere. Any value of x in the interval $[0, r]$ gives a reasonable value of y, and therefore an inscribed cylinder, so these are the only restrictions. (Note: We allow the "degenerate" cases $x = 0$, and $x = r$, so as to be able to use the shortcut for maximization on closed bounded intervals.)

Our problem is now to find the maximum value of $V(x) = 2\pi r^2 x - 2\pi x^3$ for x in the interval $[0, r]$. We search for critical numbers. We have $V'(x) = 2\pi r^2 - 6\pi x^2$ (remember that r is a constant!), so we solve:

$$2\pi r^2 - 6\pi x^2 = 0$$

$$r^2 - 3x^2 = 0$$

$$x^2 = \tfrac{1}{3}r^2$$

$$x = \pm\sqrt{\tfrac{1}{3}r^2} = \pm\sqrt{\tfrac{1}{3}} \cdot r.$$

We reject the value $x = -\sqrt{\frac{1}{3}} \cdot r$, because it is not in the interval $[0, r]$. This leaves one critical number, namely $x = \sqrt{\frac{1}{3}} \cdot r$.

Since we are maximizing a continuous function on a closed bounded interval, we can simply compare function values at the critical numbers and endpoints. We compare (using the formula $V(x) = 2\pi x(r^2 - x^2)$, because it is easier to evaluate):

$$V(0) = 0, \qquad V\left(\sqrt{\tfrac{1}{3}} \cdot r\right) = 2\pi\sqrt{\tfrac{1}{3}} \cdot r\left(r^2 - \tfrac{1}{3}r^2\right) = 2\pi\sqrt{\tfrac{1}{3}}\left(1 - \tfrac{1}{3}\right)r^3 = 2\pi\tfrac{2}{3}\sqrt{\tfrac{1}{3}} \cdot r^3,$$

and

$$V(r) = 2\pi r(r^2 - r^2) = 0.$$

Clearly $V\left(\sqrt{\frac{1}{3}} \cdot r\right)$ is the largest. Thus, the maximum volume is $V\left(\sqrt{\frac{1}{3}} \cdot r\right) = 2\pi\frac{2}{3}\sqrt{\frac{1}{3}} \cdot r^3$. (Units are not given, so we can't put them in. You can tell from this that the problem was invented by a mathematician, not a physicist.) This completes the (most direct) solution of the problem.

Te following solutions use methods we have not yet discussed, and are therefore not appropriate for this assignment. The are included for reference from later in the course.

Alternate solutions: It is also possible to use either the first or second derivative test here. Both of these will work on the open interval $(0, r)$, so for them it is not necessary to include the degenerate cases $x = 0$ and $x = r$. For both, we start at the point above where we found that $x = \sqrt{\frac{1}{3}} \cdot r$ is the only number in our interval with $V'(x) = 0$.

First derivative method: We write

$$V'(x) = 2\pi r^2 - 6\pi x^2 = 2\pi(r^2 - 3x^2).$$

(The factorization is to make it easier to determine whether V' is positive or negative.) Then $V'(x) > 0$ for $0 \leq x < \sqrt{\frac{1}{3}} \cdot r$, and $V'(x) < 0$ for $x > \sqrt{\frac{1}{3}} \cdot r$. This shows that $V(x)$ is increasing for $0 \leq x < \sqrt{\frac{1}{3}} \cdot r$ and decreasing for $x > \sqrt{\frac{1}{3}} \cdot r$. So $V\left(\sqrt{\frac{1}{3}} \cdot r\right)$ must be the largest value of $V(x)$ for x in $[0, r]$.

Second derivative method: We know $V'(x) = 2\pi r^2 - 6\pi x^2$. So $V''(x) = -12\pi x$. This is negative on the entire interval $[0, r]$, so $V(x)$ is concave down there, and any number x with $V'(x) = 0$ must give a global maximum.

Finally, let's see what happens if we solve for x instead of for y^2. From $x^2 + y^2 = r^2$, we get $x = \sqrt{r^2 - y^2}$. (Note that $x \geq 0$, so we take the positive root.) Then

$$V = 2\pi x y^2 = 2\pi\sqrt{r^2 - y^2} \cdot y^2.$$

Writing it as a function for differentiation:

$$\widetilde{V}(y) = 2\pi y^2\sqrt{r^2 - y^2}.$$

(Notice that this is a different function than the one called V above.) The restriction on y is, it turns out, the same as on x above: y must be in the interval $[0, r]$ (or $(0, r)$ if we exclude degenerate cases). We can maximize $\widetilde{V}(y)$ for y in the interval $[0, r]$ by exactly the same methods as used above, but it will be messier. (When differentiating, remember that r is a constant!) It will turn out that $\widetilde{V}(y)$ is maximized when $y = \sqrt{\frac{2}{3}} \cdot r$, which gives exactly the same cylinder as we got before.

There is a trick that can be used: since $\widetilde{V}(y) \geq 0$, it will be maximized exactly when

$$\left(\widetilde{V}(y)\right)^2 = 4\pi^2(r^2 - y^2)y^4 = 4\pi^2 r^2 y^4 - 4\pi^2 y^6$$

is maximized. This expression is much easier to differentiate, but is still a little messier than $V(x)$ from the first approach.