WORKSHEET SOLUTIONS: IMPLICIT DIFFERENTIATION

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall the chain rule: If g is differentiable at x and f is differentiable at $g(x)$, and if $h(x) = f(g(x))$ for all x (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

1. Use implicit differentiation to find $\frac{dy}{dx}$: $x^2 + y^2 = 49$. You **must** solve for $\frac{dy}{dx}$.

The above is how these problems are written in the textbook and in many WeBWorK problems. It doesn't really tell you what it should, but you need to get used to it. (Sorry: I can't change the textbook or what physicists do.) It means the following.

Assume y is a differentiable function of x given implicitly by $x^2 + [y(x)]^2 = 49$. Find $y'(x)$ in terms of x and $y(x)$.

Solution: Version 1: write y explicitly as a function of x, getting:

$$x^2 + y(x)^2 = 49$$

Differentiate, **remembering to use the chain rule**:

$$2x + 2y(x)y'(x) = 0$$

Solve for $y'(x)$:

$$\begin{aligned} 2y(x)y'(x) &= -2x \\ y'(x) &= \frac{-2x}{2y(x)} = -\frac{x}{y(x)}. \end{aligned}$$

Version 2, in physicists' notation: Differentiate, **remembering to use the chain rule**:

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}. \end{aligned}$$

2. If $y^3 = \sin(11x - y) - \sin(7)$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Solution: Let's write it with y as an explicit function $y(x)$ of x:

$$y(x)^3 = \sin(11x - y(x)) - \sin(7).$$

Differentiate both sides with respect to x, using the chain rule on both sides:

$$3[y(x)]^2 y'(x) = \cos(11x - y(x))\frac{d}{dx}(11x - y(x)) = \cos(11x - y(x))\left(11 - y'(x)\right).$$

(The derivative of $\sin(7)$ is zero because $\sin(7)$ is a constant.)

Now solve for $y'(x)$:

$$\begin{aligned} 3[y(x)]^2 y'(x) &= 11\cos(11x - y(x)) - \cos(11x - y(x))y'(x) \\ 3[y(x)]^2 y'(x) + \cos(11x - y(x))y'(x) &= 11\cos(11x - y(x)) \\ y'(x) &= \frac{11\cos(11x - y(x))}{3[y(x)]^2 + \cos(11x - y(x))}. \end{aligned}$$

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This expression can't be further simplified.

Version 2, in physicists' notation: Differentiate with respect to x, using the chain rule on both sides, just as before:

$$3y^2\frac{dy}{dx} = \cos(11x - y)\frac{d}{dx}(11x - y) = \cos(11x - y)\left(11 - \frac{dy}{dx}\right).$$

(The derivative of $\sin(7)$ is zero because $\sin(7)$ is a constant.)

Now solve for $\frac{dy}{dx}$:

$$\begin{aligned} 3y^2\frac{dy}{dx} &= 11\cos(11x - y) - \cos(11x - y)\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} + \cos(11x - y)\frac{dy}{dx} &= 11\cos(11x - y) \\ \frac{dy}{dx} &= \frac{11\cos(11x - y)}{3y^2 + \cos(11x - y)}. \end{aligned}$$

As before, this expression can't be further simplified.

Next, back to $\arctan'(x)$, from "scratch".

3. Is it more useful to differentiate both sides of the equation of functions $\arctan(\tan(x)) = x$ (valid when $-\frac{\pi}{2} < x < \frac{\pi}{2}$) or $\tan(\arctan(x)) = x$ (valid for all real x)? Remember that you will use the chain rule, and you want $\arctan'(x)$ somewhere in the answer.

Differentiate both sides of $\tan(\arctan(x)) = x$. (You can also do this as an implicit differentiation problem: $\tan(y) = x$.)

4. Carry out the differentiation from the previous step, and solve for $\arctan'(x)$.

Since $\tan(\arctan(x)) = x$ is an equation of **functions**, we can differentiate both sides with respect to x. Use the chain rule on the left:

$$\begin{aligned} \frac{d}{dx}\big(\tan(\arctan(x))\big) &= \frac{d}{dx}(x) \\ \tan'(\arctan(x))\arctan'(x) &= 1 \\ \sec^2(\arctan(x))\arctan'(x) &= 1 \\ \arctan'(x) &= \frac{1}{\sec^2(\arctan(x))}. \end{aligned}$$

Implicit differentiation method ($y(x) = \arctan(x)$):

$$\begin{aligned} \tan'(y)\frac{dy}{dx} &= 1. \\ \sec^2(y)\frac{dy}{dx} &= 1. \\ \frac{dy}{dx} = \frac{1}{\sec^2(y)} &= \frac{1}{\sec^2(\arctan(x))}. \end{aligned}$$

For the remaining solutions, see yesterday's solution sheet.