MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 5

This homework sheet is due in class on Wednesday 5 February 2025 (week 5), in class. Write answers on a separate piece of 8.5 by 11 inch paper, well organized and well labelled, with each solution starting on the left margin of the page. Or, print a 2-sides copy of this page and write on it.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

Point values as indicated, total 50 points.

1. (20 points.) A postmodern artist wants to make a frame for a painting consisting of four different exotic materials on the four sides. The material for the top costs 1 shekel per inch, the material for the bottom costs 3 shekels per inch, the material for the right hand side costs 7 shekels per inch, and the material for the left hand side costs 9 shekels per inch. The frame is to cost at most 320 shekels. Use calculus to find the width and height of the painting of largest area that can be enclosed. Include units. Verify that your solution corresponds to the largest possible area, using methods we have seen so far in the course.

Solution: Let x be the width (length of the top and bottom sides), and let y be the height (length of the right and left sides), both in inches. Let A be the area. Thus

$$A = xy. \tag{1}$$

We want to maximize this quantity. Let C be the total cost, in shekels.

We have

$$C = x + 7y + 3x + 9y.$$

(The terms are, in order, the costs of the top, right, bottom, and left fences.) We are also given $C = 320$. (There is no possible gain from not using all the money.) Combining the two formulas for C:

$$320 = C = x + 7y + 3x + 9y = 4x + 18y.$$

Solve for one of the variables, say x:

$$x = \frac{1}{4}(320 - 18y) = 80 - 4y. \tag{2}$$

Substitute this in (1) and write as a function of y:

$$A(y) = (80 - 4y)y = 80y - 4y^2.$$

We obviously must have $y \geq 0$. Also, clearly $x \geq 0$. By (2), this means $80 - 4y \geq 0$, so $y \leq 20$. Therefore we need to *maximize* $A(y) = 80y - 4y^2$ on the interval $[0, 20]$.

Find the critical points: $A'(y) = 80 - 8y$, so $A'(y) = 0$ exactly when $y = 10$. Now compare the values of A at the critical points and endpoints:

$$A(10) = (80 - 4 \cdot 10) \cdot 10 = 400, \qquad A(0) = 0, \qquad \text{and} \qquad A(20) = 0.$$

The largest is clearly $A(10)$. So take $y = 10$. By (2), this means $x = 40$. So the largest area is gotten by taking the width to be $x = 40$ inches and the height to be $y = 10$ inches.

For previous examples for Problem 2 below, see the solutions to the real Midterm 1 for 9:00 am (`https://pages.uoregon.edu/ncp/Courses/Math251_W25_Web/Midterm_1/M1_251_W2025_09_Soln.pdf`) (Problem 10), the solutions to the real Midterm 1 for 11:00 am (`https://pages.uoregon.edu/ncp/Courses/Math251_W25_Web/Midterm_1/M1_251_W2025_11.pdf`) (Problem 10), the solutions to the Midterm 1 review session worksheet (`https://pages.uoregon.edu/ncp/Courses/Math251_W25_Web/Midterm_1/M1Worksheet_Soln.pdf`) (Problem 10), the solutions to the sample Midterm 1 (`https://pages.uoregon.edu/ncp/Courses/Math251_W25_Web/Midterm_1/M1Sample_Soln.pdf`) (Problems 8 and 9), and the solutions to Written Homework 2 (`https://`

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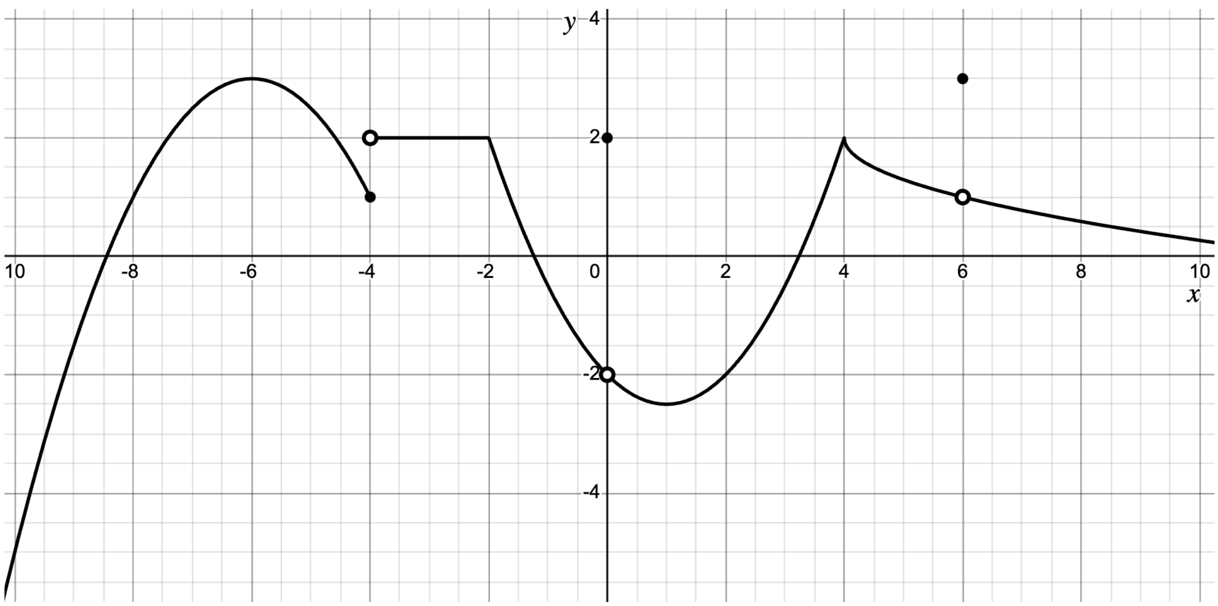
pages.uoregon.edu/ncp/Courses/Math251_W25_Web/Weekly_schedule/Week_02/WrittenHW2_Soln.pdf) (Problem 5). In the textbook, see the graphs on the following pages, and the associated discussion: 135, 140, 143, 145 (for now, the important point here is that $\lim_{x\to 2} f(x)$ does not exist), 151, 180, 182, 235–237, and 239. (We will return to one sided limits and to infinite limits.)

It is very important that you understand limits, continuity, and derivatives in terms of the pictures. Otherwise, the pictures I draw on the board during class etc. will not make sense.

2. (5 points/part) For the function $y = w(x)$ graphed below, answer the following questions.



(a) Does $\lim_{x\to -4} w(x)$ exist? If so, what is it? If not, why not?

Solution: The limit $\lim_{x\to -4} w(x)$ does not exist. Informally, there is a jump in the graph of the function at that point. Formally, $\lim_{x\to -4^+} w(x) = 2$ and $\lim_{x\to -4^-} w(x) = 1$. Since the one sided limits don't agree, the limit does not exist..

It is not true that ~~$\lim_{x\to -4} w(x) = 1$~~. That is $w(-4)$.

(b) Does $\lim_{x\to 6} w(x)$ exist? If so, what is it? If not, why not?

Solution: You can see from the graph that one can make $w(x)$ as close as one wants to 1 by requiring that x be close enough to 6 but different from 6. Therefore $\lim_{x\to 6} w(x) = 1$.

It is not true that ~~$\lim_{x\to 6} w(x) = 3$~~. That is $w(6)$.

(c) What is the largest interval containing 2 on which w is continuous? Why?

Solution: $(0, 6)$. It is clear from the graph that w is continuous on this interval: there are no breaks, jumps, or holes. However, w is not continuous at either 0 or 6: $\lim_{x\to 0} w(x) = -2$ but $w(6) = 2$, and $\lim_{x\to 6} w(x) = 1$ but $w(6) = 3$.

(d) Which of the following best describes $w'(-9)$? Why?

(1) $w'(-9)$ does not exist.
(2) $w'(-9)$ is close to 0.
(3) $w'(-9)$ is positive and not close to 0.
(4) $w'(-9)$ is negative and not close to 0.
(5) None of the above.

Solution: $w'(-9)$ is the slope of the tangent line to the graph of $y = w(x)$ at $x = -9$. You can tell from inspection that this slope is positive and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around 6. (In fact, $w'(-9) = 6$. also, it is clear that $w'(x) > 0$ when $-10 < x < -6$.)

(e) Which of the following best describes $w'(-6)$? Why?
(1) $w'(-6)$ does not exist.
(2) $w'(-6)$ is close to 0.
(3) $w'(-6)$ is positive and not close to 0.
(4) $w'(-6)$ is negative and not close to 0.
(5) None of the above.

Solution: $w'(-6)$ is the slope of the tangent line to the graph of $y = w(x)$ at $x = -6$. You can tell from inspection that the tangent line is nearly horizonal, so its slope is close to 0. So choice (2) above is correct. (In fact, for the function used, $w'(-6) = 0$.)

(f) List all points in $(-10, 10)$ at which w is not differentiable. Give reasons.

Solution: The answer is $x = -4$, $x = -2$, $x = 0$, $x = 4$, and $x = 6$. The function w is not differentiable at -4, 0, and 6, since w is not continuous at those points. Also, is not differentiable at -2 and 4, becuase there are corners in the graph at those points, so there is no tangent line.