WORKSHEET SOLUTIONS: INFINITE LIMITS

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Recall: We say $\lim_{x\to a} f(x) = \infty$ (even though ∞ is not a number) if one can force $f(x)$ to be as large as one wants by requiring that x be close enough to a. In particular, f has a vertical asymptote at $x = a$, near which the graph goes up on both sides. Example: $\lim_{x\to o} \dfrac{1}{x^2} = \infty$.

Plot this on your calculator to see, moving the screen around to show the parts of the function near $x = 0$.

Also look at the following table of values. You see that when x is close to zero (with either $x > 0$ or $x < 0$), then x^2 is positive and close to zero, and $\dfrac{1}{x^2}$ is ;positive and large.

x	x^2	$\dfrac{1}{x^2}$
1	1	1
0.1	0.01	100
0.01	0.0001	10,000
0.0001	0.00000001	100,000,000
-1	1	1
-0.1	0.01	100
-0.01	0.0001	10,000
-0.0001	0.00000001	100,000,000

Similar considerations give meanings to $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to\infty} f(x) = \pm\infty$, etc.

1. What are the following? Try computing some values if needed. After you have answered, look at calculator graphs to check.

$\lim_{x\to 0} \left(-\dfrac{1}{x^2}\right) = -\infty.$

Some values:

x	x^2	$-\dfrac{1}{x^2}$
1	1	-1
0.1	0.01	-100
0.01	0.0001	$-10,000$
0.0001	0.00000001	100,000,000
-1	1	-1
-0.1	0.01	-100
-0.01	0.0001	$-10,000$
-0.0001	-0.00000001	$-100,000,000$

When x is close to zero, then x^2 is very close to zero and positive, so $-\frac{1}{x^2}$ is very far from zero and **negative**.

$\lim_{x\to 0^-} \dfrac{1}{x} = -\infty.$

Some values:

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x	$\dfrac{1}{x}$
-1	-1
-0.1	-10
-0.01	-100
-0.0001	$-10{,}000$

When x is close to zero **and negative**, then $\frac{1}{x}$ is very far from zero and **negative**.

$\lim\limits_{x\to 2^+} \dfrac{3}{x-2} = \infty$.
Some values:

x	$x-2$	$\dfrac{3}{x-2}$
3	1	3
2.1	0.1	30
2.01	0.01	300
2.0001	0.0001	30,000

When x is close to 2 **and** $x > 2$, then $x-2$ is close to zero **and positive**, so $\frac{3}{x-2}$ is very far from zero and **positive**.

$\lim\limits_{x\to -\infty} (7x^2+1) = \infty$.
Some values:

x	x^2	$7x^2+1$
-1	1	8
-10	100	701
-100	10,000	70,001
$-10{,}000$	100,000,000	700,000,001

When x is very far from zero and negative (that is, x is negative and $|x|$ is very large), then $7x^2+1$ **positive** and very large.

Recall that ∞ and $-\infty$ are not numbers, so we can't write arithmetic operations on them. Nevertheless, some limits behave as if you could. For example, $\lim\limits_{x\to 0}\left(-\dfrac{1}{x^2}\right) = -\infty$ and $\lim\limits_{x\to 0}(x^2 - 2\cos(x)) = -2$, so $\lim\limits_{x\to 0}\left(-\dfrac{1}{x^2} + x^2 - 2\cos(x)\right) = -\infty$ and $\lim\limits_{x\to 0}\left(-\dfrac{1}{x^2}\right)(x^2 - 2\cos(x)) = \infty$. Reasons: when x is close to 0 but $x \neq 0$, then $-\frac{1}{x^2}$ is very far from zero and **negative**, while $x^2 - 2\cos(x)$ is very close to -2. So $-\frac{1}{x^2} + x^2 - 2\cos(x)$ is still very far from zero and **negative**, while $\frac{1}{x^2}(x^2 - 2\cos(x))$ is now very far from zero and **positive**.

2. Suppose now

$$\lim_{x\to 3} f(x) = \infty, \qquad \lim_{x\to 3} g(x) = \infty, \qquad \lim_{x\to 3} h(x) = 25, \qquad \lim_{x\to 3} j(x) = 0. \qquad \text{and} \qquad \lim_{x\to 3} k(x) = \frac{1}{100}.$$

What can you say about the following?

$\lim\limits_{x\to 3}(f(x)+g(x)) = \infty$. If both $f(x)$ and $g(x)$ are positive and very large, then so is $f(x)+g(x)$.

You can't say anything about $\lim_{x\to 3}(f(x)-g(x))$. As just one example, take $f(x)=\frac{1}{(x-3)^2}$ and $g(x)=\frac{1}{(x-3)^2}+3\pi^4$. Then $\lim_{x\to 3}(f(x)-g(x))=-3\pi^4$. This is an **indeterminate form**.

$\lim_{x\to 3}(f(x)-h(x))=\infty$. If $f(x)$ is positive and very large (say, much bigger than 10^6), and $h(x)$ is close to 25, then $f(x)-h(x)$ is still positive and very large, even if not quite as large (in the case above, still much bigger than 999,975).

$\lim_{x\to 3} f(x)g(x)=\infty$. If both $f(x)$ and $g(x)$ are positive and very large, then so is $f(x)g(x)$.

$\lim_{x\to 3} f(x)k(x)=\infty$. If $f(x)$ is positive and very large (say, much bigger than 100,000,000), and $k(x)$ is close to $\frac{1}{100}$, then $f(x)k(x)$ is still positive and very large, even if not quite as large (in the case above, still much bigger than 1,000,000).

You can't say anything about $\lim_{x\to 3} f(x)j(x)$. As just one example, take $f(x)=\frac{1}{(x-3)^2}$ and $j(x)=35\pi^5(x-3)^2$. Then $\lim_{x\to 3} f(x)j(x)=35\pi^5$. This is an **indeterminate form**.

3. Use these ideas to find

$$\lim_{x\to 0^+}\left(\frac{26}{x}+42x^2+7\right),\qquad \lim_{x\to 0^-}\left(\frac{26}{x}+42x^2+7\right),$$
$$\lim_{x\to \infty}\left(\frac{26}{x}+42x^2+7\right),\qquad \text{and}\qquad \lim_{x\to -\infty}\left(\frac{26}{x}+42x^2+7\right).$$

We have

$$\lim_{x\to 0^+}\frac{26}{x}=\infty,\qquad \lim_{x\to 0^-}\frac{26}{x}=-\infty,\qquad \lim_{x\to \infty}\frac{26}{x}=0,\qquad \text{and}\qquad \lim_{x\to -\infty}\frac{26}{x}=0,$$

while

$$\lim_{x\to 0^+}42x^2=0,\qquad \lim_{x\to 0^-}42x^2=0,\qquad \lim_{x\to \infty}42x^2=\infty,\qquad \text{and}\qquad \lim_{x\to -\infty}42x^2=\infty.$$

So

$$\lim_{x\to 0^+}\left(\frac{26}{x}+42x^2+7\right)=\infty,\qquad \lim_{x\to 0^-}\left(\frac{26}{x}+42x^2+7\right)=-\infty,$$
$$\lim_{x\to \infty}\left(\frac{26}{x}+42x^2+7\right)=\infty,\qquad \text{and}\qquad \lim_{x\to -\infty}\left(\frac{26}{x}+42x^2+7\right)=\infty.$$

4. Find $\displaystyle\lim_{x\to 3^+}\frac{x-7}{x-3}$.

Since $\displaystyle\lim_{x\to 3^+}(x-7)=-4$ and $\displaystyle\lim_{x\to 3^+}\frac{1}{x-3}=\infty$, we have $\displaystyle\lim_{x\to 3^+}\frac{x-7}{x-3}=-\infty$.