WORKSHEET SOLUTIONS: RELATED RATES

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Interstate 25 and Interstate 40 meet at a right angle in central Albuquerque (NM). At noon one day, Professor Greenbottle was driving north on Interstate 25 at 70 mph, and was 3 miles south of the intersection. At the same time, a Mafia hit squad was driving west on Interstate 40 at 50 mph, and was 4 miles west of the intersection. Were Professor Greenbottle and the Mafia hit squad getting closer together or further apart? At what rate? (Be sure to include the correct units.)

Steps:

- Understand the problem!
- Draw a picture if possible. (There is no picture in the solutions.) Name every quantity that varies with a **letter** (a function of time). (Here there are three of them in this problem.)
- State the information given, and what is to be found.
- Relate the variables. Choose a form of the relation that is easy to differentiate.
- Differentiate the relation, getting an equation involving quantities and their derivatives. **Be sure to use the chain rule!** In this problem, if there are less than three derivatives, you made a mistake.
- Put known values in the equation above, and solve for the quantity asked for.

Note: There is no picture in this file.

Let $a(t)$ be the distance (in miles) Professor Greenbottle's car is south of the intersection at time t (with time measured in hours), and let $b(t)$ be the distance (in miles) the hit squad is west of the intersection at time t. Let t_0 represent noon on the day in question. Let $z(t)$ be the distance (also in miles) between the two cars at time t. Then the information given says that:

$$a(t_0) = 3, \qquad a'(t_0) = -70, \qquad b(t_0) = 4, \qquad \text{and} \qquad b'(t_0) = 50.$$

(Note that $a'(t_0)$ is negative, because Professor Greenbottle's car is getting closer to the intersection. There are other possible choices for measuring distance.)

We want to find $z'(t_0)$. We know (from the Pythagorean Theorem) that $z(t)^2 = a(t)^2 + b(t)^2$. Differentiating, we get

$$2z(t)z'(t) = 2a(t)a'(t) + 2b(t)b'(t).$$

(Don't forget to use the chain rule!) Put $t = t_0$ and (for simplicity) divide by 2:

$$z(t_0)z'(t_0) = a(t_0)a'(t_0) + b(t_0)b'(t_0).$$

Next, substitute values. (Note that this can only be done *after* differentiating!) We need

$$z(t_0) = \sqrt{a(t_0)^2 + b(t_0)^2} = \sqrt{3^2 + 4^2} = 5,$$

and we then get

$$5 \cdot z'(t_0) = 3 \cdot (-70) + 4 \cdot 50 = -10.$$

Therefore $z'(t_0) = -2$, and Professor Greenbottle's car and the Mafia hit squad are getting closer to each other at 2 miles per hour. (The units are necessary!)

In physicists' notation, the equation relating the quantities is $z^2 = a^2 + b^2$. Differentiating (using the chain rule, because everything is a function of t!), we get

$$2z\frac{dz}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt},$$

so

$$z\frac{dz}{dt} = a\frac{da}{dt} + b\frac{db}{dt}.$$

Substituting values (implicitly putting $t = t_0$, and using $z = 5$ at $t = t_0$, as above):

$$5 \cdot \frac{dz}{dt} = 3 \cdot (-70) + 4 \cdot 50 = -10.$$

So $\dfrac{dz}{dt} = -2$.

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