The first problems are on estimation. To do the definite integrals on the rest of this worksheet, you will need the list of “recognizable derivatives”, and you will need to be prepared to adjust constants, simplify or otherwise modify integrands, and recognize a few cases in what the Chain Rule is relevant.

1. Here is the graph of a function \( y = f(x) \):

What are the best numbers \( r \) and \( R \) you can easily find for which you can be sure the bounds

\[ r \leq \int_{-2}^{10} f(x) \, dx \leq R \]

are correct? (If you are counting grid squares, you are doing more work than asked for.)

Solution: The graph shows that \( 1 \leq f(x) \leq 4 \) for all \( x \) in \([-2, 10]\). So

\[ 1 \cdot (10 - (-2)) \leq \int_{-2}^{10} f(x) \, dx \leq 4 \cdot (10 - (-2)), \]

that is,

\[ 12 \leq \int_{-2}^{10} f(x) \, dx \leq 48. \]

2. Here is the graph of a function \( y = g(x) \):

What are the best numbers \( r \) and \( R \) you can easily find for which you can be sure the bounds

\[ r \leq \int_{-2}^{10} g(x) \, dx \leq R \]
are correct? (If you are counting grid squares, you are doing more work than asked for.)

Solution: The graph shows that \(-1 \leq g(x) \leq 2\) for all \(x\) in \([-2, 10]\). So

\[-1 \cdot (10 - (-2)) \leq \int_{-2}^{10} g(x) \, dx \leq 2 \cdot (10 - (-2)),\]

that is,

\[-12 \leq \int_{-2}^{10} g(x) \, dx \leq 24.\]

3. Find the following definite integrals. Give exact values. (Don’t use a calculator.) Answers must be simplified.

Solution:

\[
\int_{0}^{2} x^3 \, dx = \left. \frac{x^4}{4} \right|_{0}^{2} = \frac{2^4}{4} = 4.
\]

\[
\int_{2}^{11} 3 \sin(x) \, dx = \left. -3 \cos(x) \right|_{2}^{11} = -3 \cos(11) + 3 \cos(2).
\]

\[
\int_{1}^{2} (x^4 - 5x^2) \, dx = \int_{1}^{2} x^4 \, dx - \int_{1}^{2} 3x^2 \, dx = \left. \frac{x^5}{5} \right|_{1}^{2} - \left. \frac{5x^3}{3} \right|_{1}^{2} = \frac{2^5}{5} - \frac{1^5}{5} - \left( \frac{5 \cdot 2^3}{3} - \frac{5 \cdot 1^3}{3} \right) = \frac{32}{5} - \frac{31}{5} = 1.
\]

There is no point in carrying out the fraction subtraction.

\[
\int_{1}^{5} \frac{x^2 + 6}{x} \, dx = \int_{1}^{5} \left( \frac{x^2}{x} - \frac{6}{x} \right) \, dx = \int_{1}^{5} x \, dx - \int_{1}^{5} \frac{6}{x} \, dx
\]

\[
= \left. \frac{x^2}{2} \right|_{1}^{5} - 6 \ln(x) \bigg|_{1}^{5} = \frac{5^2}{2} - \frac{1^2}{2} - [6 \ln(5) - 6 \ln(1)]
\]

\[= 12 - 6 \ln(5).\]

(We used \(\ln(1) = 0\). This simplification is required.)

\[
\int_{2}^{6} \cos(2x) \, dx = \frac{1}{2} \sin(2x) \bigg|_{2}^{6} = \frac{1}{2} [\sin(12) - \sin(4)].
\]

\[
\int_{0}^{10} xe^{-x^2} \, dx = -\frac{1}{2} e^{-x^2} \bigg|_{0}^{10} = -\frac{1}{2} [e^{-100} - e^0] = \frac{1}{2} [1 - e^{-100}].
\]

(We used \(e^0 = 1\). This simplification is required.)

\[
\int_{-1}^{2} \frac{x^2}{1 + x^6} \, dx = \frac{1}{3} \arctan(x^3) \bigg|_{-1}^{2} = \frac{1}{3} [\arctan(8) - \arctan(-1)]
\]

\[= \frac{1}{3} [\arctan(8) + \arctan(1)].\]