1. Here are the graphs of the curves $y^2 = x$ and $y = x^3$.

We want to find the area of the region between these two graphs.

a. First, find the coordinates of the two points at which the graphs intersect. (What equations do you need to solve?)

Solution: Square the second equation to get $y^2 = x^6$. Combine with the first equation: $x = x^6$ So

$$x^6 - x = 0$$
$$x^5(x - 1) = 0$$
$$x = 0 \text{ or } x = 1.$$ If $x = 0$ then $y = 0$ and if $x = 1$ then $y = 1$. So the points are $(0, 0)$ and $(1, 1)$.

b. Now, consider a very narrow vertical rectangle contained in this region, at $x = \frac{1}{2}$ and with width $\Delta x$. Approximately what is the area of this rectangle? (You will need to use the formulas for the functions. You should not simplify your answer.)

Solution: Set $f(x) = \sqrt{x}$ and $g(x) = x^3$. Then the height is very close to $f\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)$ and the width is $\Delta x$, so the area is

$$(f\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)) \Delta x = \left(\sqrt{\frac{1}{2}} - \left(\frac{1}{2}\right)^3\right) \Delta x.$$  

c. Next, consider a very narrow vertical rectangle contained in this region, at $x = \frac{1}{3}$ and with width $\Delta x$. Approximately what is the area of this rectangle? (You should not simplify your answer.)

Date: 7 February 2018.
**Solution:** Set \( f(x) = \sqrt{x} \) and \( g(x) = x^3 \). Then the height is very close to \( f\left(\frac{1}{4}\right) - g\left(\frac{1}{4}\right) \) and the width is \( \Delta x \), so the area is

\[
(f\left(\frac{1}{4}\right) - g\left(\frac{1}{4}\right)) \Delta x = \left(\sqrt{\frac{1}{4}} - \left(\frac{1}{4}\right)^3\right) \Delta x.
\]

d. Suppose now that we cover the region with many very narrow vertical rectangles, and add up their areas. Does this look like a Riemann sum for a definite integral? Which one?

**Solution:** Set \( f(x) = \sqrt{x} \) and \( g(x) = x^3 \). Then it is

\[
\int_0^1 (f(x) - g(x)) \, dx = \int_0^1 (\sqrt{x} - x^3) \, dx.
\]

e. Find the area of the region.

**Solution:**

\[
\int_0^1 (\sqrt{x} - x^3) \, dx = \int_0^1 \sqrt{x} \, dx - \int_0^1 x^3 \, dx = \frac{2}{3} x^{3/2} \bigg|_0^1 - \frac{1}{4} x^4 \bigg|_0^1 = \left(\frac{2}{3} - 0\right) - \left(\frac{1}{4} - 0\right) = \frac{7}{12}.
\]

2. Here are the graphs of the functions \( y = -x^2 \) and \( y = \sqrt{3 + x^4} - 3 \).

![Graphs of functions](image)

Set up an integral which gives the area between these two curves. (You will need to solve for the \( x \)-coordinates of the points at which they intersect.) Don’t try to evaluate the integral.

**Solution:** We first need the \( x \)-coordinates of the points where the curves intersect. We solve:

\[
\sqrt{3 + x^4} - 3 = -x^2
\]
\[
\sqrt{3 + x^4} = 3 - x^2 \\
3 + x^4 = (\sqrt{3 + x^4})^2 = (3 - x^2)^2 = 9 - 6x^2 + x^4 \\
0 = 6x^2 - 6 = 6(x - 1)(x + 1) \\
x = -1 \quad \text{or} \quad x = 1.
\]

Now set \( f(x) = -x^2 \) and \( g(x) = \sqrt{3 + x^4} - 3 \). The area is
\[
\int_{-1}^{1} (f(x) - g(x)) \, dx = \int_{-1}^{1} (-x^2 - (\sqrt{3 + x^4} - 3)) \, dx \\
= \int_{-1}^{1} (3 - x^2 - \sqrt{3 + x^4}) \, dx.
\]

3. Here are the graphs of the curves \( y = e^{x/4} \), \( y = 2e^{x/4} \), \( y = 1 \), and \( y = 2 \).

\[ \text{a. Find the area of the region bounded by those two curves. (Using the methods from above, it will be the sum of two integrals.)} \]

\textit{Solution:} We first need the \( x \)-coordinates of the points where the curves intersect. There are four of them, given by the equations
\[
e^{x/4} = 1, \quad e^{x/4} = 2, \quad 2e^{x/4} = 1, \quad \text{and} \quad 2e^{x/4} = 2.
\]
The first and last equations both have the solution \( x = 4 \ln(1) = 0 \). The second equation has the solution \( x = 4 \ln(2) \), and the third equation has the solution \( x = 4 \ln\left(\frac{1}{2}\right) = -4 \ln(2) \).

Now
\[
\int_{-4 \ln(2)}^{0} (2e^{x/4} - 1) \, dx + \int_{0}^{4 \ln(2)} (2 - e^{x/4}) \, dx \\
= \left(8e^{x/4} - x\right) \bigg|_{-4 \ln(2)}^{0} + \left(2x - 4e^{x/4}\right) \bigg|_{0}^{4 \ln(2)} \\
= 8 \exp(0) - 0 - 8 \exp(-4 \ln(2)/4) + (-4 \ln(2)) \\
+ 2(4 \ln(2)) - 4 \exp(4 \ln(2)/4) - 2 \cdot 0 + 4 \exp(0) \\
= 8 - 0 - 4 - 4 \ln(2) + 8 \ln(2) - 8 - 0 + 4 = 4 \ln(2).
\]
b. Now find the area of the same region by thinking of the areas of thin horizontal rectangles with height \( \Delta y \).

\textit{Solution:} Write the equations of the two curves as
\[ x = 4 \ln(y) \quad \text{and} \quad x = 4 \ln(y/2) = 4 \ln(y) - 4 \ln(2). \]
Then the area is
\[ \int_{1}^{2} \left[ 4 \ln(y/2) - (4 \ln(y) - 4 \ln(2)) \right] dy = \int_{1}^{2} 4 \ln(2) \, dy = 4 \ln(2). \]