MATH 252 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4.

This homework sheet is due in class on Friday 9 February 2018 (week 5).

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and use correct notation. (See the web page on notation.)

1. (10 points.) Do the following derivation (which was skimmed over in class on Tuesday 6 February 2018):

\[
\int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \ln \left( |\sec(x) + \tan(x)| \right) \right] + C.
\]

(Recall the method: integrate by parts, then use a trigonometric identity to get something involving \( \int \sec^3(x) \, dx \) again, and solve for \( \int \sec^3(x) \, dx \).)

Solution: Integrate by parts, taking

\[ u(x) = \sec(x), \quad v'(x) = \sec^2(x) \quad u'(x) = \sec(x) \tan(x), \quad \text{and} \quad v(x) = \tan(x). \]

Thus,

\[
\int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec(x) \tan(x) \, dx = \sec(x) \tan(x) - \int \sec(x) \tan^2(x) \, dx.
\]

Apply the trigonometric identity \( \tan^2(x) = \sec^2(x) - 1 \), getting

\[
\int \sec(x) \tan^2(x) \, dx = \int \sec(x) \left( \sec^2(x) - 1 \right) \, dx = \int \sec^3(x) \, dx - \int \sec(x) \, dx,
\]

so

\[
\int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec^3(x) \, dx + \int \sec(x) \, dx.
\]

Therefore

\[
2 \int \sec^3(x) \, dx = \sec(x) \tan(x) + \int \sec(x) \, dx,
\]

so

\[
\int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \int \sec(x) \, dx \right].
\]

Now recall \( \int \sec(x) \, dx \) from class, and put it in:

\[
\int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \ln \left( |\sec(x) + \tan(x)| \right) \right] + C.
\]

2. (10 points.) Find the area between the parabola \( x = y^2 \) and the line \( y = 2 - x \).

Solution: We first need to know the shape of the region. Here is a picture:
We find the points where the curves intersect by solving the simultaneous equations \( x = y^2 \) and \( y = 2 - x \). The second says \( x = 2 - y \). Substituting in the first gives \( 2 - y = y^2 \), or
\[
0 = y^2 + y - 2 = (y - 1)(y + 2).
\]
So the solutions are \( y = 1 \) and \( y = -2 \). Using the equation \( x = 2 - y \) to find the corresponding values of \( x \), we see that the points are \( (1, 1) \) and \( (4, -2) \).

The region extends from \( y = -2 \) to \( y = 1 \). Using the equation \( x = 2 - y \) for the line, we see that a horizontal slice at \( y \) has length \((2 - y - y^2)\). So the area is
\[
\int_{-2}^{1} (2 - y - y^2) \, dy = \left[ 2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1} = 2 - \frac{1^2}{2} - \frac{1^3}{3} - \left(2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right) = 2 - \frac{1}{2} - \frac{1}{3} + 4 - \frac{8}{3} = \frac{9}{2}.
\]

Alternate solution: Find the points where the two curves intersect as in the first solution. The region extends from \( x = 0 \) to \( x = 4 \). For \( 0 \leq x \leq 1 \), using the equations \( y = \sqrt{x} \) and \( y = -\sqrt{x} \) for the two branches of the parabola, we see that a vertical slice at \( x \) has length \( 2\sqrt{x} \). For \( 1 \leq x \leq 4 \), using the equation \( y = -\sqrt{x} \) for the lower branch of the parabola, we see that a vertical slice at \( x \) has length \( 2 - x + \sqrt{x} \). So the area is
\[
\int_{0}^{1} 2\sqrt{x} \, dx + \int_{1}^{4} (2 - x + \sqrt{x}) \, dx = \left( \frac{4}{3}x^{3/2} \right) \bigg|_{0}^{1} + \left( 2x - \frac{1}{2}x^2 - \frac{2}{3}x^{3/2} \right) \bigg|_{1}^{4} = \frac{4}{3} + \left( 8 - 8 + \frac{16}{3} \right) - \left( \frac{1}{2} - \frac{1}{2} + \frac{2}{3} \right) = \frac{9}{2}.
\]

The first solution was certainly easier, but you should know how to do the second as well.

3. (10 points.) Find
\[
\int_{0}^{1} \frac{1}{4x^2 + 4x + 17} \, dx.
\]
Hint: Complete the square in the denominator to find a substitution which reduces this integral to \( \int \frac{1}{u^2 + 1} \, du \).

Solution: Write
\[
\int \frac{1}{4x^2 + 4x + 17} \, dx = \int \frac{1}{(2x + 1)^2 + 4} \, dx = \int \frac{1}{4\left(\left(\frac{1}{2}(2x + 1)\right)^2 + 1\right)} \, dx = \int \left( \frac{1}{4^2} \right) \frac{1}{\left(\frac{1}{2}(2x + 1)\right)^2 + 1} \, dx.
\]
Now make the substitution \( u = \frac{1}{4}(2x + 1) \), so \( du = \frac{1}{2} \, dx \), or \( dx = 2 \, du \). Thus,
\[
\int \left( \frac{1}{4^2} \right) \frac{1}{\left(\frac{1}{2}(2x + 1)\right)^2 + 1} \, dx = \int \left( \frac{1}{8} \right) \frac{1}{u^2 + 1} \, du = \frac{1}{8} \arctan(u) + C = \frac{1}{8} \arctan\left( \frac{1}{4}(1 + 2x) \right) + C.
\]
4. (5 points/part.) Use a suitable substitution to convert each of the following integrals into one involving only integer powers of the standard trigonometric functions. Do not evaluate the resulting integrals.

   a. \( \int (4 - x^2)^{-3/2} \, dx \), for \(-2 < x < 2\).

   Solution: Since 1 - \sin^2(\theta) = \cos^2(\theta), we take \( x = 2 \sin(\theta) \), with \( \theta \) in \((\frac{\pi}{2}, \frac{\pi}{2})\), so \( dx = 2 \cos(\theta) \, d\theta \). Thus (using \cos(\theta) > 0 when we take the square root)
   \[
   \int (1 - x^2)^{-3/2} \, dx = \int 2 \cos(\theta) (4 - 4\sin^2(\theta))^{-3/2} \, d\theta = \int 2 \cos(\theta) 4^{-3/2} (\cos^2(\theta))^{-3/2} \, d\theta = \int \frac{1}{4 \cos^2(\theta)} \, d\theta.
   \]

   b. \( \int (1 + 4x^2)^{3/2} \, dx \).

   Solution: Since \( \tan^2(\theta) + 1 = \sec^2(\theta) \), we take \( x = \frac{1}{2} \tan(\theta) \), with \( \theta \) in \((\frac{\pi}{2}, \frac{\pi}{2})\), so \( dx = \frac{1}{2} \sec^2(\theta) \, d\theta \). Thus (using \sec(\theta) > 0 when we take the square root)
   \[
   \int (1 + 4x^2)^{3/2} \, dx = \int \frac{1}{2} \sec^2(\theta) (1 + \tan^2(\theta))^{3/2} \, d\theta = \int \frac{1}{2} \sec^2(\theta) (\sec^2(\theta))^{3/2} \, d\theta = \int \frac{1}{2} \sec^5(\theta) \, d\theta.
   \]

   c. \( \int \frac{1}{\sqrt{x^2 - 1}} \, dx \), for \( x > 1 \).

   Solution: Since \( \sec^2(\theta) - 1 = \tan^2(\theta) \), we take \( x = \sec(\theta) \), with \( \theta \) in \((0, \frac{\pi}{2})\), so \( dx = \tan(\theta) \sec(\theta) \, d\theta \). Thus (using \tan(\theta) > 0 when we take the square root)
   \[
   \int \frac{1}{\sqrt{x^2 - 1}} \, dx = \int \frac{\tan(\theta) \sec(\theta)}{\sqrt{\sec^2(\theta) - 1}} \, d\theta = \int \frac{\tan(\theta) \sec(\theta)}{\sqrt{\tan^2(\theta)}} \, d\theta = \int \sec(\theta) \, d\theta.
   \]