1. (10 points.) Do the following derivation (which was skimmed over in class on Tuesday 6 February 2018):

\[ \int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \ln \left( |\sec(x) + \tan(x)| \right) \right] + C. \]

(Recall the method: integrate by parts, then use a trigonometric identity to get something involving \( \int \sec^3(x) \, dx \) again, and solve for \( \int \sec^3(x) \, dx \).)

Solution: Integrate by parts, taking
\[ u(x) = \sec(x), \quad v'(x) = \sec^2(x) \quad u'(x) = \sec(x) \tan(x), \quad \text{and} \quad v(x) = \tan(x). \]

Thus,
\[ \int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec(x) \tan(x) \, \tan(x) \, dx = \sec(x) \tan(x) - \int \sec(x) \tan^2(x) \, dx. \]

Apply the trigonometric identity \( \tan^2(x) = \sec^2(x) - 1 \), getting
\[ \int \sec(x) \tan^2(x) \, dx = \int \sec(x) \left( \sec^2(x) - 1 \right) \, dx = \int \sec^3(x) \, dx - \int \sec(x) \, dx, \]

so
\[ \int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec^3(x) \, dx + \int \sec(x) \, dx. \]

Therefore
\[ 2 \int \sec^3(x) \, dx = \sec(x) \tan(x) + \int \sec(x) \, dx, \]

so
\[ \int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \int \sec(x) \, dx \right]. \]

Now recall \( \int \sec(x) \, dx \) from class, and put it in:
\[ \int \sec^3(x) \, dx = \frac{1}{2} \left[ \sec(x) \tan(x) + \ln \left( |\sec(x) + \tan(x)| \right) \right] + C. \]

2. (10 points.) Find the area between the parabola \( x = y^2 \) and the line \( y = 2 - x \).

Solution: We first need to know the shape of the region. Here is a picture:
We find the points where the curves intersect by solving the simultaneous equations \( x = y^2 \) and \( y = 2 - x \). The second says \( x = 2 - y \). Substituting in the first gives \( 2 - y = y^2 \), or

\[
0 = y^2 + y - 2 = (y - 1)(y + 2).
\]

So the solutions are \( y = 1 \) and \( y = -2 \). Using the equation \( x = 2 - y \) to find the corresponding values of \( x \), we see that the points are \((1, 1)\) and \((4, -2)\).

The region extends from \( y = -2 \) to \( y = 1 \). Using the equation \( x = 2 - y \) for the line, we see that a horizontal slice at \( y \) has length \( 2 - y - y^2 \). So the area is

\[
\int_{-2}^{1} (2 - y - y^2) \, dy = \left[ 2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1} = 2 - \frac{1^2}{2} - \frac{1^3}{3} - \left( 2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3} \right) = 2 - \frac{1}{2} - \frac{1}{3} + 4 + \frac{8}{3} = \frac{9}{2}.
\]

Alternate solution: Find the points where the two curves intersect as in the first solution. The region extends from \( x = 0 \) to \( x = 4 \). For \( 0 \leq x \leq 1 \), using the equations \( y = \sqrt{x} \) and \( y = -\sqrt{x} \) for the two branches of the parabola, we see that a vertical slice at \( x \) has length \( 2\sqrt{x} \). For \( 1 \leq x \leq 4 \), using the equation \( y = -\sqrt{x} \) for the lower branch of the parabola, we see that a vertical slice at \( x \) has length \( 2 - x + \sqrt{x} \). So the area is

\[
\int_{0}^{1} 2\sqrt{x} \, dx + \int_{1}^{4} (2 - x + \sqrt{x}) \, dx = \left[ \frac{4}{3} x^{3/2} \right]_{0}^{1} + \left[ 2x - \frac{1}{2} x^2 + \frac{2}{3} x^{3/2} \right]_{1}^{4} = \frac{4}{3} + \left( 8 - 8 + \frac{16}{3} \right) - \left( 2 - \frac{1}{2} + \frac{2}{3} \right) = \frac{9}{2}.
\]

The first solution was certainly easier, but you should know how to do the second as well.

3. (10 points.) Find

\[
\int \frac{1}{4x^2 + 4x + 17} \, dx.
\]

Hint: Complete the square in the denominator to find a substitution which reduces this integral to \( \int \frac{1}{u^2 + 1} \, du \).

Solution: Write

\[
\int \frac{1}{4x^2 + 4x + 17} \, dx = \int \frac{1}{(2x + 1)^2 + 4} \, dx = \int \frac{1}{4((\frac{1}{2})(2x + 1))^2 + 1} \, dx = \int \left( \frac{1}{4^2} \right) \frac{1}{(\frac{1}{4})(2x + 1))^2 + 1} \, dx.
\]

Now make the substitution \( u = \frac{1}{4}(2x + 1) \), so \( du = \frac{1}{2} \, dx \), or \( dx = 2 \, du \). Thus,

\[
\int \left( \frac{1}{4^2} \right) \frac{1}{(\frac{1}{4})(2x + 1))^2 + 1} \, dx = \int \left( \frac{1}{8} \right) \frac{1}{u^2 + 1} \, du = \frac{1}{8} \arctan(u) + C = \frac{1}{8} \arctan\left( \frac{1}{4}(1 + 2x) \right) + C.
\]
4. (5 points/part.) Use a suitable substitution to convert each of the following integrals into one involving only integer powers of the standard trigonometric functions. Do not evaluate the resulting integrals.

a. \( \int (4 - x^2)^{-3/2} \, dx \), for \(-2 < x < 2\).

**Solution:** Since \( 1 - \sin^2(\theta) = \cos^2(\theta) \), we take \( x = 2 \sin(\theta) \), with \( \theta \) in \( \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \), so \( dx = 2 \cos(\theta) \, d\theta \). Thus (using \( \cos(\theta) > 0 \) when we take the square root)

\[
\int (1 - x^2)^{-3/2} \, dx = \int 2 \cos(\theta) (4 - 4 \sin^2(\theta))^{-3/2} \, d\theta \\
= \int 2 \cos(\theta) 4^{-3/2} (\cos^2(\theta))^{-3/2} \, d\theta = \int \frac{1}{4 \cos^2(\theta)} \, d\theta.
\]

b. \( \int (1 + 4x^2)^{3/2} \, dx \).

**Solution:** Since \( \tan^2(\theta) + 1 = \sec^2(\theta) \), we take \( x = \frac{1}{2} \tan(\theta) \), with \( \theta \) in \( \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \), so \( dx = \frac{1}{2} \sec^2(\theta) \, d\theta \). Thus (using \( \sec(\theta) > 0 \) when we take the square root)

\[
\int (1 + 4x^2)^{3/2} \, dx = \int \frac{1}{2} \sec^2(\theta) (1 + \tan^2(\theta))^{3/2} \, d\theta \\
= \int \frac{1}{2} \sec^2(\theta) (\sec^2(\theta))^{3/2} \, d\theta = \frac{1}{2} \int \sec^5(\theta) \, d\theta.
\]

c. \( \int \frac{1}{\sqrt{x^2 - 1}} \, dx \), for \( x > 1 \).

**Solution:** Since \( \sec^2(\theta) - 1 = \tan^2(\theta) \), we take \( x = \sec(\theta) \), with \( \theta \) in \( (0, \frac{\pi}{2}) \), so \( dx = \tan(\theta) \sec(\theta) \, d\theta \). Thus (using \( \tan(\theta) > 0 \) when we take the square root)

\[
\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \int \tan(\theta) \sec(\theta) \frac{1}{\sqrt{\sec^2(\theta) - 1}} \, d\theta \\
= \int \tan(\theta) \sec(\theta) \frac{1}{\sqrt{\tan^2(\theta)}} \, d\theta = \int \sec(\theta) \, d\theta.
\]