1. The region between the curve $y = 8 - 6x + \frac{1}{2}x^3$ and the $x$-axis, to the right of the $y$-axis, and to the left of $x = 2$ (where the curve intersects the $x$-axis) is rotated around the $y$-axis to give a spiky solid.

Here are five pictures, two of the solid, one of the graph of $y = 8 - 6x + \frac{1}{2}x^3$, one of the graph of $y = 8 - 6|x| + \frac{1}{2}|x|^3$, and one of the solid with a cylindrical shell (see part (a) below):

We want to find its volume.

a. Consider a thin cylindrical shell with inner radius 1, thickness $\Delta x$, and height equal to the height of the solid at $x = 1$. (It looks something like in the picture on the right side of the second row, but the cylindrical shell shown there has outer radius 1.) What is its volume? Expand your answer in powers of $\Delta x$.

b. Your answer to Part (a) should have terms with $\Delta x$ and $(\Delta x)^2$. What is the linear approximation (linear in $\Delta x$)?

c. Repeat Parts (a) and (b) at $x = \frac{1}{2}$. Again, expand in powers of $\Delta x$ and cancel as appropriate, but don’t simplify further.

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d. Suppose now that we cover the solid with many very thin cylindrical shells with appropriate heights, and add up the linear approximations to their volumes. Does this look like a Riemann sum for a definite integral? Which one?

e. Find the volume of the solid.

2. The region between the lines $x = 1$, $x = 2$, and $y = x - 1$, and the curve $y = (x - 1)^3$, is rotated around the $y$-axis to give a solid which looks sort of like a piece of pipe which is thicker at one end than the other. Here are three pictures, one of the solid, one of the graphs of $y = x - 1$ and $y = (x - 1)^3$, and one showing this graph and the part symmetrically opposite it:

We want to find its volume.

a. Consider a thin cylindrical shell with inner radius $\frac{3}{2}$, thickness $\Delta x$, and which just fits inside the solid at $x = \frac{3}{2}$. What is its volume? Expand your answer in powers of $\Delta x$.

b. Your answer to Part (a) should have terms with $\Delta x$ and $(\Delta x)^2$. What is the linear approximation (linear in $\Delta x$)?

c. Suppose now that we cover the solid with many very thin cylindrical shells with appropriate heights, and add up the linear approximations to their volumes. Does this look like a Riemann sum for a definite integral? Which one?

d. Set up an integral which gives the volume of the solid. (Don’t calculate it, but be sure to get the right limits of integration.)
3. The linear density of a piece of wire in the mass per unit length. Thus, a particular kind of wire might have a linear density of 2 grams per centimeter, which means that a 1 cm piece would have a mass of 2 grams, a 14 cm piece would have a mass of 28 grams, and (since we are mathematicians here) a −3 cm piece would have a mass of −6 grams.

The wire might not have constant linear density. For example, it might be thicker in some places than others. (Perhaps the machine used to manufacture it wasn’t working properly.)

Suppose you have a piece of wire which is 30 cm long and whose linear density \( x \) cm from its left end is \( 2 + \sin(\pi x/20) \) grams per centimeter. We want to find the total mass of this wire.

a. For small positive \( \Delta x \), consider the piece of wire between 10 and 10 + \( \Delta x \) cm from the left end of the wire. Give an approximation to the mass of this piece which is good when \( \Delta x \) is small.

b. For small positive \( \Delta x \), consider the piece of wire between 3 and 3 + \( \Delta x \) cm from the left end of the wire. Give an approximation to the mass of this piece which is good when \( \Delta x \) is small.

c. Suppose we divide the wire into many short pieces, each of length \( \Delta x \), and add up the approximations to their masses. Does this look like a Riemann sum for a definite integral? Which one?

d. Find the total mass of the wire.

4. Find the volume of the solid in Problem 2.