WORKSHEET: BASIC MECHANICS OF DIFFERENTIAL EQUATIONS

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1. Consider the differential equation

\[ y'(x) - 2y(x) = 2e^{4x} \]

on \((-\infty, \infty)\). Verify that the function

\[ y(x) = e^{4x} \]

satisfies this equation. That is, check that if the function \( y \) is defined by (2) for all real numbers \( x \), then the equation (1) holds for all real numbers \( x \).

**Solution:** For every real number \( x \), we have

\[ y'(x) - 2y(x) = 4e^{4x} - 2e^{4x} = 2e^{4x}. \]

2. Consider the differential equation

\[ f'(x) - 2f(x) = 2 - 2x^2 \]

on \((-\infty, \infty)\). Verify that the function

\[ f(x) = x^2 \]

does not satisfy this equation. That is, if the function \( f \) is defined by (4) for all real numbers \( x \), find some real number \( x \) such that the equation (3) is false.

**Solution:**

\[ f'(x) - 2f(x) = \frac{d}{dx}(x^2) - 2x^2 = 2x - 2x^2. \]

If we put \( x = 0 \), we get \( f'(x) - 2f(x) = 0 \), but the right hand side of (3) is equal to 2. Since \( 0 \neq 2 \), this shows that (3) does not hold.

Many other choices of \( x \) will work. However, taking \( x = 1 \) will not work: the equation (3) does hold for this choice of \( x \).

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3. Find all solutions to the differential equation \( f'(x) = x \sin(x^2) \) on \((-\infty, \infty)\). That is, find all functions \( f \) defined on \((-\infty, \infty)\) such that \( f'(x) = x \sin(x^2) \) for all real numbers \( x \).

Solution: The problem is just a fancy way of asking for all antiderivatives of \( x \sin(x^2) \). So, using the substitution \( u = x^2 \):

\[
\int x \sin(x^2) \, dx = -\frac{1}{2} \cos(x^2) + C,
\]

for an arbitrary real constant \( C \).

4. Verify that the function \( y(x) = 0 \) for all real \( x \) is a solution to the differential equation \( y'(x) = 6y(x) \) on \((-\infty, \infty)\). Then find a nonzero solution. Can you find a family of solutions with one free parameter? Can you find a solution \( y \) such that \( y(0) = 2 \)?

Solution: If \( y(x) = 0 \) for all \( x \), then \( y'(x) = 0 \) for all \( x \) and \( 6y(x) = 0 \) for all \( x \).

The most obvious nonzero solution is \( y(x) = e^{6x} \).

The most obvious family of solution is \( y(x) = C e^{6x} \) for an arbitrary real constant \( C \). (These are all the solutions. Use the method of Problem 8 below.)

If \( y(0) = 2 \) and \( y(x) = C e^{6x} \) for all \( x \), then \( 2 = y(0) = C e^{6\cdot0} = C \). So \( y(x) = 2 e^{6x} \) works.

5. Consider the differential equation \( g''(t) - 5g'(t) + g(t) = t^2 \) on \((-\infty, \infty)\). Find a real number \( a \) such that \( g(t) = t^2 + at + 48 \) satisfies this equation.

Solution: Put the formula for \( g \) into the equation:

\[
g''(t) - 5g'(t) + g(t) - t^2
= \frac{d^2}{dt^2}(t^2 + at + 48) - 5 \frac{d}{dt}(t^2 + at + 48) + t^2 + at + 48 - t^2
= \frac{d}{dt}(2t + a) - 5(2t + a) + at + 48
= 2 - 10t - 5a + at + 48 = 5(10 - a) + (a - 10)t.
\]

If \( a = 10 \), then this expression is zero for all real numbers \( t \).
6. Consider the differential equation \( y''(t) - 7y'(t) + 12y(t) = 0 \) on \((-\infty, \infty)\). Find all real numbers \( k \) such that the function \( y(t) = e^{kt} \) is a solution to this equation.

**Solution:** We have \( y'(t) = ke^{kt} \) and \( y''(t) = k^2e^{kt} \). So, for all real numbers \( t \),
\[
y''(t) - 7y'(t) + 12y(t) = k^2e^{kt} - 7ke^{kt} + 12e^{kt} = (k^2 - 7k - 4)e^{kt} = (k - 3)(k - 4)e^{kt}.
\]
This expression is supposed to be zero for all real numbers \( t \). If \((k - 3)(k - 4) = 0\), that is, if \( k = 3 \) or \( k = 4 \), then clearly \((k - 3)(k - 4)e^{kt} = 0\) for all real numbers \( t \). So these two values of \( k \) work. For any other value of \( k \), we have \((k - 3)(k - 4) \neq 0\). Taking \( t = 0 \), we get
\[
y''(0) - 7y'(0) + 12y(0) = (k - 3)(k - 4)e^0 = (k - 3)(k - 4) \neq 0,
\]
so \( y(t) = e^{kt} \) is not a solution.

7. For a fixed constant \( k > 0 \), find at least one nonzero solution to the differential equation \( x''(t) = -kx(t) \).

**Solution:** Remembering that \( \sin''(t) = -\sin(t) \), we try a function of the form \( x(t) = \sin(\omega t) \) for some real number \( \omega \). Rewrite the equation as \( x''(t) + kx(t) = 0 \), and substitute this function for \( x \), getting
\[
0 = x''(t) + kx(t) = \frac{d}{dt} (\omega \cos(\omega t)) + k \sin(\omega t) = -\omega^2 \sin(\omega t) + k \sin(\omega t).
\]
If \( \omega = \sqrt{k} \), then the last expression is zero for all real \( t \). So \( x(t) = \sin(\sqrt{k} t) \) is a nonzero solution.

Remark: The most general solution is
\[
x(t) = a \sin(\sqrt{k} t) + b \cos(\sqrt{k} t)
\]
for arbitrary (real) constants \( a \) and \( b \).

8. Consider the differential equation \( f'(x) = 3f(x) \) on \((-\infty, \infty)\). For every real constant \( C \), the function \( f(x) = Ce^{3x} \) is a solution to this equation. Are these all of the solutions?

Suppose \( f \) is an arbitrary solution to this differential equation. Use the differential equation to find
\[
\frac{d}{dx} (e^{-3x} f(x))
\]
in terms of $x$ and $f(x)$. (The expression $f'(x)$ should not appear.) Then simplify your answer as much as possible. What does this tell you about $e^{-3x}f(x)$? What does this tell you about $f(x)$?

**Solution:** Using $f'(x) = 3f(x)$ at the second step, we get, for all $x$,
\[
\frac{d}{dx} \left( e^{-3x}f(x) \right) = -3e^{-3x}f(x) + e^{-3x}f'(x) = -3e^{-3x}f(x) + e^{-3x} \cdot 3f(x) = 0.
\]
Therefore $x \mapsto e^{-3x}f(x)$ is a constant function, that is, there is a constant $C$ such that $e^{-3x}f(x) = C$ for all real $x$. But then $f(x) = Ce^{3x}$ for all real $x$.

**Bonus question 1.** We want to find a solution the differential equation $f'(x) + 2f(x) = 24e^{6x}$ on $(-\infty, \infty)$. Multiply both sides of it by $e^{2x}$. Do you recognize the left hand side as being the derivative of some function $H(x)$ (whose definition will involve $f(x)$)? Find $H(x)$ by integration, and use your answer to find $f(x)$.

**Solution:** Multiplying both sides by $e^{2x}$ gives
\[
e^{2x}f'(x) + 2e^{2x}f(x) = 24e^{8x}.
\]
Write this as
\[
e^{2x}f'(x) + \frac{d}{dx} \left( e^{2x}f(x) \right) = 24e^{8x}.
\]
If $H(x) = e^{2x}f(x)$, then the left hand side is $H'(x)$. So $H'(x) = 24e^{8x}$. Therefore there is a constant $C$ such that $H(x) = 3e^{8x} + C$. So
\[
f(x) = e^{-2x}H(x) = e^{-2x} \left( 3e^{8x} + C \right) = 3e^{6x} + Ce^{-2x}.
\]

**Bonus question 2.** Find a function $f$ defined on $(-\infty, \infty)$ such that $f'(x) - f(x) = e^x$ for all real $x$ and $f(0) = 12$.

**Solution:** Multiplying both sides of the equation by $e^{-x}$ gives
\[
\frac{d}{dx} \left( e^{-x}f(x) \right) = e^{-x}f'(x) - e^{-x}f(x) = 1.
\]
Therefore there is a constant $C$ such that $e^{-x}f(x) = x + C$ for all real $x$. So $f(x) = xe^x + Ce^x$ for all real $x$. Putting $x = 0$ and using $f(0) = 12$, we get $12 = f(0) = C$. So $f(x) = xe^x + 12e^x$ works.