1. A population of fire-breathing monsters on the planet Yuggxth grows at a rate proportional to the existing population.

(a) Write a differential equation which describes this situation. (It will may contain one or more unknown constants, which you will have to specify.)

Solution: Let \( P(t) \) be the population at time \( t \). There is a constant \( k \) (the growth rate) such that \( P'(t) = kP(t) \).

(b) What is the general solution to the differential equation in part (a)?

Solution: You can get this by the method of separable equations, but this equation is common enough that you should recognize the solution:

\[
P(t) = Ce^{kt}
\]

with \( k \) as above and for an arbitrary constant \( C \).

(c) Suppose there were 5 million fire-breathing monsters in 1950 and 7 million fire-breathing monsters in 2010. Predict the number of fire-breathing monsters in 2030.

Solution: We measure \( t \) in years since 1950. (This is not the only possible choice.) We measure the population in millions. (Again. this is not the only possible choice.) Combine the information given with \( P(t) = Ce^{kt} \) to get

\[
5 = P(0) = Ce^{k\cdot0} = C \quad \text{and} \quad 7 = P(60) = Ce^{k\cdot60}
\]

to get, first, \( C = 5 \), then

\[
7 = 5e^{60k}.
\]

Solving this last equation gives

\[
k = \frac{1}{60} \ln \left( \frac{7}{5} \right).
\]

We have \( P(80) \), which is

\[
P(7) = Ce^{k\cdot80} = 5 \exp \left( \frac{80}{60} \ln \left( \frac{7}{5} \right) \right) = 5 \exp \left( \frac{4}{3} \ln \left( \frac{7}{5} \right) \right).
\]

(d) In an alternate universe, the population of fire-breathing monsters triples every 172 years. Find the growth constant (or growth rate). How long does it take the population to double?

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Solution: We measure $t$ in years. If there are $P_0$ of them at time zero, there will be $3P_0$ of them at time 172. The solution with $P_0$ of them at time zero is $P(t) = P_0e^{kt}$, so we have

$$3P_0 = P(172) = P_0e^{k\cdot172}.$$  

We can assume $P_0 > 0$, so this equation becomes

$$3 = e^{k\cdot172},$$

so

$$k = \frac{\ln(3)}{172}.$$  

Now the general solution is

$$P(t) = P_0 \exp\left(\frac{\ln(3)}{172}t\right)$$  

for a constant $P_0$. We need to find $t_0$ such that

$$2P_0 = P(t_0) = P_0 \exp\left(\frac{\ln(3)}{172}t_0\right).$$  

We can assume $P_0 > 0$, so this equation becomes

$$\exp\left(\frac{\ln(3)}{172}t_0\right) = 2.$$  

So

$$\frac{\ln(3)}{172}t_0 = \ln(2),$$

whence

$$t_0 = \frac{172\ln(2)}{\ln(3)}.$$  

The population doubles after $172\ln(2)/\ln(3)$ years. (The units are required.)

3. A hot (or cold) object is put out in a room whose temperature is $T_0$. Newton’s Law of Cooling, approximately valid if the initial temperature difference is not too big and the object is much smaller than the room, states that the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the temperature of the room.

(a) Write a differential equation which describes this situation. (It will may contain one or more unknown constants, which you will have to specify.)
Solution: Let $T(t)$ be the temperature of the object at time $t$. There is a constant $k$ (the proportionality constant) such that $T'(t) = k(T(t) - T_0)$.

(b) You should have one constant in your solution to part (a). Is it positive or negative? Why?

Solution: In the form $T'(t) = k(T(t) - T_0)$ for some constant $k$, the constant should be negative. Reason: if $T(t) > T_0$, the object is hotter than its surroundings, and so its temperature should be decreasing.

(c) Find the general solution to the differential equation in part (a). (You can solve it using separation of variables, but it is easier to write the equation in terms of the difference between the temperature of the object and the ambient temperature.)

Solution: Set $D(t) = T(t) - T_0$.

(d) A cold glass of milk is put out on a table. Its temperature is 5 deg. C, and after 2 hours its temperature is 20 deg. C. The room temperature is 25 deg. C. When did the temperature of the glass of milk reach 15 deg. C?

Solution: We measure $T$ in deg. C, and $t$ in hours since the milk was put out. We have $T_0 = 24$, so

$$5 = T(0) = 25 + Ce^{k \cdot 0} = 25 + C$$

and

$$20 = T(2) = 5 + Ce^{k \cdot 2} = 25 + Ce^{2k}.$$  
The first equation implies $C = -20$. So the second equation says

$$20 = 25 - 20e^{2k},$$

so $e^{2k} = \frac{1}{4}$, so

$$k = \frac{1}{2} \ln \left( \frac{1}{4} \right) = -\frac{1}{2} \ln(4) = \ln(2).$$

Now the solution to the differential equation is

$$T(t) = 25 - 20 \exp(-\ln(2)t)$$

for all $t$. We need to find $t_0$ such that $T(t_0) = 15$. That is,

$$15 = T(t_0) = 25 - 20 \exp(-\ln(2)t_0).$$
So \(-20 \exp(-\ln(2)t_0) = -10\), which says \(\exp(-\ln(2)t_0) = \frac{1}{2}\). So \(-\ln(2)t_0 = \ln\left(\frac{1}{2}\right) = -\ln(2)\), whence \(t = 1\). The temperature 15 deg. C was reached after one hour. (The units are required.)

3. A tank of water contains 1000 liters of a solution of potassium iodide in water. A solution of 3 grams of potassium iodide per liter of water is pumped into the tank, at a rate of 4 liters per hour, and water is overflowing from the tank (and contaminating the room containing the tank) at the same rate. The contents of the tank are kept well mixed.

Let \(K(t)\) be the number of grams of potassium iodide in the tank at time \(t\), with \(t\) measured in hours.

There are two contributions to the rate of change of \(K(t)\): one from the water flowing into the tank and another from water leaving the tank.

(a) What is the contribution to \(K'(t)\) from from the water flowing into the tank? Include units.

Solution: There are 4 liters per hour of water flowing into the tank, and each liter contains 3 grams of potassium iodide, so this contribution increases the amount of potassium iodide in the tank by 12 grams per hour.

(b) What is the contribution to \(K'(t)\) from from the water leaving the tank? Include units.

Solution: There are 4 liters per hour of water leaving into the tank, and each liter contains \(K(t)/1000\) grams of potassium iodide, so this contribution decreases the amount of potassium iodide in the tank by \(4K(t)/1000\) grams per hour.

(c) Write a differential equation for \(K(t)\).

Solution: We have to combine the two contributions:

\[
K'(t) = 12 - \frac{4K(t)}{1000}.
\]

(d) Find the general solution to the differential equation in part (c). (You can solve it using separation of variables, but this equation is very similar to the one for Newton’s Law of Cooling, and the method used there works here too.)
Solution: Rewrite the equation as

\[ K'(t) = \left( \frac{4}{1000} \right) \left( \frac{3000}{4} - K(t) \right). \]

Set

\[ L(t) = K(t) - \frac{3000}{4}. \]

Then

\[ L'(t) = K'(t) = - \left( \frac{4}{1000} \right) \left( \frac{3000}{4} - K(t) \right) = - \left( \frac{4}{1000} \right) L(t). \]

so there is a constant \( C \) such that

\[ L(t) = C \exp \left( -\frac{4}{1000} t \right) \]

for all \( t \). So

\[ K(t) = L(t) + \frac{3000}{4} = C \exp \left( -\frac{4}{1000} t \right) + \frac{3000}{4} \]

for all \( t \).