MATH 252 (PHILLIPS): PRACTICE WRITTEN HOMEWORK ON SETTING UP DIFFERENTIAL EQUATIONS.

This homework sheet is not officially part of the course grade. Anything turned in by 11:00 am Monday 19 March 2018 (or perhaps later) will be graded (for your information only), and returned on Tuesday (I hope at or soon after the review session).

There is an example in Problem 4 in written homework 8. Here is another one.

Example. A chemical reaction in a solution involves combining a molecule of compound A and a molecule of compound B to produce a molecule of compound C. The reaction rate is proportional to the product of the concentrations of compound A and compound B. (This is a constant times the rate at which molecules of compound A collide with molecules of compound B.) Suppose that the initial concentration of compound A is \(a_0\) the initial concentration of compound B is \(b_0\), and compound C is not present.

Set up a differential equation which models the amount of compound C in the solution as a function of time. Be sure to state what your variables mean. You will have two unknown constants in your equation; state that they are unknown constants, and state whether they are positive or negative.

Set up a differential equation which models the proportion of the population which has heard the rumor as a function of time. Be sure to state what your variables mean. You will have one unknown constant in your equation; state that it is unknown constant, and state whether it is positive or negative.

Solution: Let \(c(t)\) be the concentration of compound C in the solution at time \(t\). There is only one contribution to the rate of change of the concentration, namely some constant times the concentration of compound A times the concentration of compound B. The concentration of of compound B at time \(t\) is \(a_0 - c(t)\), since \(a_0\) is the original concentration and \(x(t)\) is how much of it has been converted to compound C. Similarly, The concentration of of compound B at time \(t\) is \(b_0 - c(t)\).

We thus get the differential equation

\[ c'(t) = k(a_0 - c(t))(b_0 - c(t)) \]

for some constants \(k > 0\).

Problems:

1. A fire-breathing monster is trapped in a room with volume 300 cubic meters. It consumes 2 cubic meters of oxygen per day. Fresh air containing 25% oxygen enters the room at 12 cubic meters per day. The air in the room is well mixed (partly by the escape attempts of the monster), and 12 cubic meters of this air leaves the room per day.

Set up a differential equation which models the amount of oxygen in the room as a function of time. Be sure to state what your variables mean.

2. The spread of a rumor in a large homogeneous population is modelled as follows. People meet at random, and each time somebody who has not heard the rumor meets somebody who has, there is a fixed probability that the person who has heard the rumor passes it in. Thus, the rate at which new people hear the rumor is proportional to the product of the number of people who have already heard it and the number of people who have not yet heard it.

Set up a differential equation which models the proportion of the population which has heard the rumor as a function of time. Be sure to state what your variables mean. You will have one unknown constant in your equation; state that it is unknown constant, and state whether it is positive or negative.

Date: 19 March 2018.
(It is easier to use the proportion of the population which has heard the rumor, because the proportion of the population which has not heard the rumor is 1 minus the proportion which has heard it. If you use the number of people instead, you need to introduce a constant for the total population.)

3. The spread of a new evangelical religion in a large homogeneous population is modelled as follows. People meet at random, and each time a nonbeliever meets a believer, there is a fixed probability that believer manages to convert the nonbeliever to the new religion. Thus, the rate at which nonbelievers become believers is proportional to the product of the number of nonbelievers and the number of believers. In addition, believers abandon the religion at a rate proportional to the number of believers.

Set up a differential equation which models the proportion of believers in the population as a function of time. Be sure to state what your variables mean. You will have two unknown constants in your equation; state that they are unknown constants, and state whether they are positive or negative.

(It is easier to use the proportion of believers, because the proportion of nonbelievers is 1 minus the proportion which has heard it. If you use the number of believers instead, you need to introduce a constant for the total population.)

4. A chemical reaction in a solution involves combining two molecules of compound A to produce a molecule of compound B. The rate at which two molecules of compound A are converted to one of compound B is proportional to the rate of collisions between molecules of compound A, which is proportional to the square of the concentration of compound A. In addition, molecules of compound A are added to the solution at a constant rate.

Set up a differential equation which models the amount of compound A in the solution as a function of time. Be sure to state what your variables mean. You will have two unknown constants in your equation; state that they are unknown constants, and state whether they are positive or negative.

5. Atoms of carbon-14 are radioactive, and decay at a rate proportional to their number. They are also created at a constant rate by collisions of cosmic rays of atoms of nitrogen-14.

Set up a differential equation which models the amount of carbon-14 present as a function of time. Be sure to state what your variables mean. You will have two unknown constants in your equation; state that they are unknown constants, and state whether they are positive or negative.

6. A region of the planet Glorppth is inhabited by dragons and goblins. The main source of food of the dragons is by eating goblins. We assume both dragons and goblins are roughly uniformly distributed, and meet at random. So the number of encounters is proportional to the product of the number of dragons and the number of goblins. When a dragon meets a goblin, there is a fixed probability that the dragon catches and eats the goblin.

In the absence of dragons, the population of goblins would grow at a rate proportional to the existing population. Also, however, they get eaten at a rate proportional to the product of the number of dragons and the number of goblins. In the absence of goblins, the dragons die of starvation at a rate proportional to the existing population. Starvation is counteracted by eating goblins, so there is a positive contribution to the growth of the population of dragons proportional to the product of the number of dragons and the number of goblins.

Set up differential equations which model the populations of goblins and dragons as a function of time. (Both populations will appear in each equation.) Be sure to state what your variables mean. You will have four unknown constants in your equation; state that they are unknown constants, and state whether they are positive or negative.