Math 252       Solutions to the Sample Questions for Midterm 2

At least 80% of the points on the real exam will be modifications of problems from the problems below, homework problems (particularly written homework), worksheet problems, and problems from the sample and real Midterms 0 and 1. Note, though, that the exact form of the functions to be integrated could vary substantially, and the methods required to do them might occur in different combinations.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation in the book, in handouts, in files posted on the course website, and on the blackboard; use it. The right notation will help you get the mathematics right, and incorrect notation will lose points.

The exam is closed book, except for a 3 × 5 file card, written on both sides, but readable without a magnifying glass. The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.

Problems.

1. (10 points.) Evaluate the definite integral \[ \int_{1}^{\infty} \frac{\ln(y)}{y} \, dy, \] or else show that it does not converge.

   Solution: This is an improper integral. We first find the indefinite integral using the substitution \( u = \ln(y), \) so \( du = \frac{1}{y} \, dy. \) Thus

   \[ \int \frac{\ln(y)}{y} \, dy = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln(y))^2 + C. \]

   Now

   \[ \int_{1}^{\infty} \frac{\ln(y)}{y} \, dy = \frac{1}{2} (\ln(y))^2 \bigg|_{1}^{\infty} = \lim_{y \to \infty} \left( \frac{1}{2} (\ln(y))^2 \right) = \infty. \]

   Therefore the integral \( \int_{1}^{\infty} \frac{\ln(y)}{y} \, dy \) diverges.

2. (10 points.) Evaluate the definite integral \[ \int_{0}^{\infty} \frac{e^{3y}}{4 + e^{6y}} \, dy, \] or else show that it does not converge.
Solution: This is an improper integral. We first find the indefinite integral using the substitution $u = e^{3y}$, so $du = 3e^{3y} \, dy$ and $e^{3y} \, dy = \frac{1}{3} \, du$:

$$
\int \frac{e^{3y}}{4 + e^{6y}} \, dy = \int \frac{1}{4 + (e^{3y})^2} \, e^{3y} \, dy = \int \left( \frac{1}{3} \right) \frac{1}{4 + u^2} \, du = \int \left( \frac{1}{12} \right) \frac{1}{1 + \frac{1}{4}u^2} \, du.
$$

Now substitute $w = \frac{1}{2}u$, so $dw = \frac{1}{2} \, du$ and $du = 2 \, dw$. Thus

$$
\int \left( \frac{1}{12} \right) \frac{1}{1 + \frac{1}{4}u^2} \, du = \int \left( \frac{1}{12} \right) \frac{2}{1 + w^2} \, dw
= \frac{1}{6} \arctan(w) + C = \frac{1}{6} \arctan \left( \frac{1}{2}u \right) + C = \frac{1}{6} \arctan \left( \frac{1}{2}e^{3y} \right) + C.
$$

Now

$$
\int_{0}^{\infty} \frac{e^{3y}}{1 + e^{2y}} \, dy = \frac{1}{6} \arctan \left( \frac{1}{2}e^{3y} \right) \bigg|_{0}^{\infty} = \lim_{y \to \infty} \left( \frac{1}{6} \arctan \left( \frac{1}{2}e^{3y} \right) - \frac{1}{6} \arctan \left( \frac{1}{2}e^{0} \right) \right).
$$

Since $\lim_{y \to \infty} \frac{1}{2}e^{3y} = \infty$ and $\lim_{y \to \infty} \arctan(y) = \frac{\pi}{2}$, we have $\lim_{y \to \infty} \arctan \left( \frac{1}{2}e^{3y} \right) = \frac{\pi}{2}$. Therefore

$$
\int \frac{e^{3y}}{1 + e^{2y}} \, dy = \frac{1}{6} \left( \frac{\pi}{2} \right) - \frac{1}{6} \arctan \left( \frac{1}{2} \right) = \frac{\pi}{12} - \frac{1}{6} \arctan \left( \frac{1}{2} \right).
$$

Evaluation of $\arctan(1)$ and the corresponding simplification are required.

3. (12 points) Evaluate the definite integral $\int_{-1}^{1} \frac{e^{x}}{e^{x} - 1} \, dx$, or else show that it does not converge.

Solution: Since $\lim_{x \to 0} \frac{e^{x}}{e^{x} - 1}$ does not exist, this is an improper integral. We must test $\int_{0}^{1} \frac{e^{x}}{e^{x} - 1} \, dx$ and $\int_{-1}^{0} \frac{e^{x}}{e^{x} - 1} \, dx$ separately. The integral $\int_{-1}^{1} \frac{e^{x}}{e^{x} - 1} \, dx$ converges if and only if both of these other integrals converge.

We first find the indefinite integral using the substitution $u = e^{x} - 1$, so $du = e^{x} \, dx$. Thus

$$
\int \frac{e^{x}}{e^{x} - 1} \, dx = \int \frac{1}{u} \, du = \ln(|u|) + C = \ln(|e^{x} - 1|) + C.
$$

Now

$$
\int_{0}^{1} \frac{e^{x}}{e^{x} - 1} \, dx = \ln(|e^{x} - 1|) \bigg|_{0}^{1} = \lim_{x \to 0^{+}} \left( \ln(|e^{1} - 1|) - \ln(|e^{x} - 1|) \right).
$$

We have $\lim_{x \to 0^{+}} |e^{x} - 1| = 0$, with $|e^{x} - 1| > 0$ far all $x$, and $\lim_{x \to 0^{+}} \ln(t) = -\infty$, so $\lim_{x \to 0^{+}} \ln(|e^{x} - 1|) = -\infty$. Since $\ln(|e^{1} - 1|)$ is a constant, it follows that $\lim_{x \to 0^{+}} \left( \ln(|e^{1} - 1|) - \ln(|e^{x} - 1|) \right) = \infty$. So $\int_{0}^{1} \frac{e^{x}}{e^{x} - 1} \, dx$ diverges. Therefore $\int_{-1}^{1} \frac{e^{x}}{e^{x} - 1} \, dx$ diverges.

It is just as good to work with $\int_{-1}^{0} \frac{e^{x}}{e^{x} - 1} \, dx$: essentially the same reasoning as above shows that this integral is $-\infty$. 
4. (8 points.) Determine whether or not the improper integral \( \int_{5}^{\infty} \frac{2 + 7 \arctan(12t)}{t} \, dt \) converges.

*Solution:* For \( t \geq 5 \) we have \( 7 \arctan(12t) \geq 0 \). So \( 0 \leq \frac{2}{t} \leq \frac{2 + 7 \arctan(12t)}{t} \). Now

\[
\int_{5}^{\infty} \frac{2}{t} \, dt = 2 \ln(t) \bigg|_{5}^{\infty} = \lim_{t \to \infty} 2(\ln(t) - \ln(5)) = \infty.
\]

So \( \int_{5}^{\infty} \frac{2}{t} \, dt \) diverges. Therefore the Comparison Test shows that \( \int_{5}^{\infty} \frac{2 + 7 \arctan(12t)}{t} \, dt \) diverges.

5. (8 points.) Determine whether or not the improper integral \( \int_{0}^{\infty} \frac{e^{-2y}}{1 + y^4} \, dy \) converges.

*Solution:* For \( y \geq 0 \) we have \( 1 + y^4 \geq 1 \). So \( 0 \leq \frac{e^{-2y}}{1 + y^4} \leq e^{-2y} \). Now

\[
\int_{0}^{\infty} e^{-2y} \, dy = -\frac{1}{2} e^{-2y} \bigg|_{0}^{\infty} = \lim_{y \to \infty} \left( -\frac{1}{2} e^{-2y} + \frac{1}{2} e^{-0} \right) = 0 + \frac{1}{2} = \frac{1}{2}.
\]

So \( \int_{0}^{\infty} e^{-2y} \, dy \) converges. Therefore \( \int_{0}^{\infty} \frac{e^{-2y}}{1 + y^4} \, dy \) converges by the Comparison Test.

*Alternate solution:* For \( y \geq 1 \) we have \( 1 + y^4 \geq 1 + y^2 \) and \( e^{-2y} \leq 1 \). So \( 0 \leq \frac{e^{-2y}}{1 + y^4} \leq \frac{1}{1 + y^2} \).

Now

\[
\int_{0}^{\infty} \frac{1}{1 + y^2} \, dy = \arctan(y) \bigg|_{0}^{\infty} = \lim_{y \to \infty} \left( \arctan(y) - \arctan(0) \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.
\]

So \( \int_{0}^{\infty} \frac{1}{1 + y^2} \, dy \) converges. Therefore \( \int_{0}^{\infty} \frac{e^{-2y}}{1 + y^4} \, dy \) converges by the Comparison Test.

6. (8 points.) Determine whether or not the improper integral \( \int_{0}^{2} \frac{9 + 2 \sin(1/x)}{x^2} \, dx \) converges.

*Solution:* For all \( t \neq 0 \) we have \( \sin(1/x) \geq -1 \), so \( 9 + 2 \sin(1/x) \geq 7 \). So \( 0 \leq \frac{9}{x^2} \leq \frac{9 + 2 \sin(1/x)}{x^2} \).

Now

\[
\int_{0}^{2} \frac{7}{x^2} \, dx = -\frac{7}{x} \bigg|_{0}^{\infty} = \lim_{x \to 0^+} \left( -\frac{7}{2} + \frac{7}{x} \right) = \infty.
\]

So \( \int_{0}^{2} \frac{7}{x^2} \, dx \) diverges. Therefore the Comparison Test shows that \( \int_{0}^{2} \frac{9 + 2 \sin(1/x)}{x^2} \, dx \) diverges.
Alternate solution: We first find the indefinite integral.

\[ \int \frac{9}{x^2} \, dx = -\frac{9}{x} + C. \]

For the other part, using the substitution \( u = \frac{1}{x} \), so \( du = -\frac{1}{x^2} \, dx \). Thus

\[ \int \frac{2 \sin(1/x)}{x^2} \, dx = \int 2u \, du = u^2 + C = -\cos \left( \frac{1}{x} \right) + C_2. \]

So (combining the constants of integration)

\[ \int \frac{9 + 2 \sin(1/x)}{x^2} \, dx = -\frac{9}{x} - \cos \left( \frac{1}{x} \right) + C. \]

Now

\[ \int_0^2 \frac{9 + 2 \sin(1/x)}{x^2} \, dx = \left( -\frac{9}{x} - \cos \left( \frac{1}{x} \right) \right) \bigg|_0^2 \]

\[ = \lim_{x \to 0^+} \left( -\frac{9}{x} - \cos \left( \frac{1}{x} \right) + \frac{9}{\frac{1}{2}} + \cos \left( \frac{1}{x} \right) \right). \]

The terms \(-\frac{9}{2}\) and \(-\cos \left( \frac{1}{2} \right)\) are constants, we have \( \lim_{x \to 0^+} \frac{9}{x} = \infty \), and \(-1 \leq \cos \left( \frac{1}{x} \right) \leq 1 \) for all \( x \neq 0 \). Therefore

\[ \lim_{x \to 0^+} \left( -\frac{9}{2} - \cos \left( \frac{1}{2} \right) + \frac{9}{x} + \cos \left( \frac{1}{x} \right) \right) = \infty. \]

So the improper integral \( \int_0^2 \frac{9 + 2 \sin(1/x)}{x^2} \, dx \) diverges.

7. (12 points.) Find the area of the region bounded by the curves \( y^2 = x + 1 \), and \( y = (x + 1)^3 \).

(It may help to draw a picture of the region first.)

Solution: Here is a picture of the region:

We need to know where the curves intersect. We must solve the simultaneous equations

\[ y^2 = x + 1 \quad \text{and} \quad y = (x + 1)^3. \]
Substituting the second equation in the first gives \((x+1)^3 = x + 1\), or \((x+1)^6 - (x+1) = 0\), which implies 

\[(x + 1)((x + 1)^5 - 1) = 0.\]

So \(x + 1 = 0\) or \((x + 1)^5 - 1 = 0\). The first says \(x = -1\), which implies \(y = 0\). The second says \((x + 1)^5 = 1\), so \(x + 1 = 1\), so \(x = 0\), implies \(y = 1\). So the intersection points are \((-1, 0)\) and \((0, 1)\).

We now use vertical slices, that is, we integrate with respect to \(x\). This requires rewriting the equation \(y^2 = x + 1\) as \(y = (x + 1)^{1/2}\). (We don’t use \(y = -(x + 1)^{1/2}\) since we know \(y \geq 0\).) Therefore the area is 

\[
\int_{-1}^{0} \left((x + 1)^{1/2} - (x + 1)^3\right) \, dx = \left(\frac{2}{3}(x + 1)^{3/2} - \frac{1}{4}(x + 1)^4\right)\bigg|_{-1}^{0} = \frac{2}{3} - \frac{1}{4} - (0 - 0) = \frac{5}{12}.
\]

**Alternate solution:** We use the same picture and computation of the intersection points as in the first solution, but we use horizontal slices, that is, we integrate with respect to \(y\). This requires rewriting the equation \(y^2 = x + 1\) as \(x = y^2 - 1\) and rewriting the equation \(y = (x + 1)^3\) as \(x = y^{1/3} - 1\). Therefore the area is 

\[
\int_{0}^{1} \left((y^{1/3} - 1) - (y^2 - 1)\right) \, dy = \int_{0}^{1} (y^{1/3}y^2) \, dy = \left(\frac{3}{4}y^{4/3} - \frac{1}{3}y^3\right)\bigg|_{-1}^{0} = \frac{3}{4} - \frac{1}{3} - (0 - 0) = \frac{5}{12}.
\]

8. (15 points.) Find the area of the region bounded by the curves \(y = -\frac{1}{4}x^2\), \(y = x\), and \(y = 3 - 2x\), and with \(0 \leq x \leq 2\). (It may help to draw a picture of the region first.)

**Solution:** Here is a picture of the region:

![Diagram](image.png)

We will use vertical slices. We need the \(x\)-coordinates of the points where the curves intersect. We must solve three sets of simultaneous equations

\[
\begin{align*}
(1) & \quad y = -\frac{1}{4}x^2 \quad \text{and} \quad y = x, \\
(2) & \quad y = x \quad \text{and} \quad y = 3 - 2x,
\end{align*}
\]
and
\[ y = -\frac{1}{4}x^2 \quad \text{and} \quad y = 3 - 2x. \]
The only solution to (1) has \( x = 0 \), and the only solution to (2) has \( x = 1 \). The equations (3) imply \( x^2 - 8x + 12 = 0 \), which has the solutions \( x = 2 \) and \( x = 6 \). We ignore \( x = 6 \) since we are supposed to assume that \( 0 \leq x \leq 2 \).

We need two integrals because the area consists of two regions bounded by different upper lines for \( x \leq 1 \) and \( x \geq 1 \). Thus the area is
\[
\int_{0}^{1} \left( x - \left( -\frac{1}{4}x^2 \right) \right) \, dx + \int_{1}^{2} \left( (3 - 2x) - \left( -\frac{1}{4}x^2 \right) \right) \, dx \\
= \left. \left( \frac{x^3}{12} + \frac{x^2}{2} \right) \right|_{0}^{1} + \left. \left( \frac{x^3}{12} - x^2 + 3x \right) \right|_{1}^{2} = \frac{31}{6}.
\]

Alternate solution (sketch): We use the same picture as in the first solution, but we use horizontal slices, that is, we integrate with respect to \( y \). The same computation as before also gives the \( y \) coordinates of the intersection points, which are 0 for (1), 1 for (1),, and \(-1\) for (1). We also have to rewrite all of the equations to give \( x \) in terms of \( y \). This gives
\[
\int_{-1}^{0} \left( \frac{1}{2} (y - 3) - 2\sqrt{-y} \right) \, dy + \int_{0}^{1} \left( \frac{1}{2} (y - 3) - y \right) \, dy.
\]

9. (5 points.) Set up an integral which represents the length of the curve \( y = 3e^{x^2} \) between \( x = 4 \) and \( x = 67 \). Don’t try to evaluate the integral.

Solution: For the graph of a differentiable function \( f \) on an interval \([a, b]\), we have seen that its length is \( \int \sqrt{1 + [f'(x)]^2} \, dx \). In this case, that is
\[
\int_{4}^{67} \sqrt{1 + (6xe^{x^2})^2} \, dx = \int_{4}^{67} \sqrt{1 + 36x^2e^{2x^2}} \, dx.
\]
(The simplification is required.)

10. (12 points.) Find the exact value of the length of the graph of the function \( f(x) = \ln(\cos(x)) \) for \( 0 \leq x \leq \frac{\pi}{4} \).

Solution: We need to calculate, using \( \sec(x) > 0 \) for \( 0 \leq x \leq \frac{\pi}{4} \) at the third step,
\[
\int_{0}^{\pi/4} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{\pi/4} \sqrt{1 + (\tan(x))^2} \, dx = \int_{0}^{\pi/4} \sqrt{\sec^2(x)} \, dx = \int_{0}^{\pi/4} \sec(x) \, dx \\
= \ln \left| \sec(x) + \tan(x) \right| \bigg|_{0}^{\pi/4} \\
= \ln \left| \sec(\pi/4) + \tan(\pi/4) \right| - \ln \left| \sec(0) + \tan(0) \right| = \ln \left( \sqrt{2} + 1 \right).
\]
11. (15 points.) Find the exact value of the length of the graph of the function \( f(x) = \frac{1}{2}(e^x + e^{-x}) \) for \( 1 \leq x \leq 3 \).

**Solution:** We need to calculate

\[
\int_1^3 \sqrt{1 + [f'(x)]^2} \, dx = \int_1^3 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2} \, dx
\]

\[
= \int_1^3 \sqrt{1 + \frac{1}{4}(e^x)^2 - \frac{1}{2} + \frac{1}{4}(e^{-x})^2} \, dx = \int_1^3 \frac{1}{2}(e^x + e^{-x}) \, dx
\]

\[
= \frac{1}{2}(e^x - e^{-x}) \bigg|_1^3 = \frac{1}{2}(e^3 - e^{-3}) - \frac{1}{2}(e - e^{-1}).
\]

12. (17 points.) Find the exact value of the length of the graph of the function \( f(x) = 3 + \frac{1}{2}x^2 \) for \(-2 \leq x \leq 3\).

**Solution (outline):** We need to calculate

\[
\int_{-2}^3 \sqrt{1 + [f'(x)]^2} \, dx = \int_{-2}^3 \sqrt{1 + x^2} \, dx.
\]

Use the substitution \( x = \tan(\theta) \). Then \( dx = (\sec(\theta))^2 \, d\theta \). We have

\[
\int_{\arctan(-2)}^{\arctan(3)} \sqrt{1 + (\tan(\theta))^2} \, (\sec(\theta))^2 \, d\theta = \int_{\arctan(-2)}^{\arctan(3)} \sec^3(\theta) \, d\theta.
\]

(Rest of solution to be filled in later.) We calculate \( \int_a^b \sec^3(\theta) \, d\theta \) when \( -\frac{\pi}{2} < a < b < \frac{\pi}{2} \).

We will need the integral

\[
\int \sec(x) \, dx = \ln(|\sec(x) + \tan(x)|) + C.
\]

This doesn’t come easily from any of the standard methods we have seen, so you just need to know it. Integrate by parts, taking

\[
u(\theta) = \sec(\theta), \quad v'(\theta) = \sec^2(\theta) \quad u'(\theta) = \sec(\theta) \tan(\theta), \quad \text{and} \quad v(\theta) = \tan(\theta).
\]
This gives
\[
\int_a^b \sec^3(\theta) \, d\theta = \sec(\theta) \tan(\theta) \bigg|_a^b - \int_a^b \tan^2(\theta) \sec(\theta) \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) \bigg|_a^b - \int_a^b (\sec^2(\theta) - 1) \sec(\theta) \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) \bigg|_a^b - \int_a^b \sec^3(\theta) \, d\theta + \int_a^b \sec(\theta) \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) \bigg|_a^b - \int_a^b \sec^3(\theta) \, d\theta + \ln(|\sec(\theta) \tan(\theta)|) \bigg|_a^b.
\]
Therefore
\[
2 \int_a^b \sec^3(\theta) \, d\theta = \sec(\theta) \tan(\theta) \bigg|_a^b + \ln(|\sec(\theta) + \tan(\theta)|) \bigg|_a^b.
\]
So
\[
\int_a^b \sec^3(\theta) \, d\theta = \frac{1}{2} \sec(\theta) \tan(\theta) \bigg|_a^b + \frac{1}{2} \ln(|\sec(\theta) + \tan(\theta)|) \bigg|_a^b.
\]
Now put \(a = \arctan(-2)\) and \(b = \arctan(3)\). We have
\[
\sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + x^2.
\]
Since \(-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}\), it follows that \(\sec(\arctan(x)) > 0\), so \(\sec(\arctan(x)) = \sqrt{1 + x^2}\). Therefore
\[
\int_{-2}^{3} \sqrt{1 + x^2} \, dx = \frac{1}{2} \sec(\theta) \tan(\theta) \bigg|_{\arctan(-2)}^{\arctan(3)} + \frac{1}{2} \ln(|\sec(\theta) + \tan(\theta)|) \bigg|_{\arctan(-2)}^{\arctan(3)}
\]
\[
= \frac{3\sqrt{10} + \ln(3 + \sqrt{10}) + 2\sqrt{5} - \ln(\sqrt{5} - 2)}{2}.
\]
Note: The calculation shows that
\[
\int \sqrt{1 + x^2} \, dx = \frac{1}{2} x \sqrt{1 + x^2} + \frac{1}{2} \ln(x + \sqrt{1 + x^2}) + C.
\]

13. (8 points.) A postmodern monument consists of a flat slab of concrete in the shape of a circle of radius 8 meters. It is to be painted with a mixture of pale orange paint and pale blue paint, with the proportions of the two colors varying across the concrete, but using two liters of paint per square meter of surface. At distance \(y\) meters north of the center, the concrete is
painted with $2e^{-y^2}$ liters of pale orange paint per square meter and $2 - 2e^{-y^2}$ liters of pale blue paint per square meter. Set up an integral which represents the total amount of pale orange paint needed for this monument. Include an explanation. Don’t try to evaluate the integral.

Solution: Consider a thin slice of the monument at distance $y$ meters north of the center, for $-8 \leq y \leq 8$, and with width $\Delta y$. It has length $\sqrt{8-y^2}$ and width $\Delta y$, so its area is $\sqrt{8-y^2} \cdot \Delta y$. Therefore it requires $2e^{-y^2} \cdot \sqrt{8-y^2} \cdot \Delta y$ liters of pale orange paint. So the total amount of pale orange paint is

$$\int_{-8}^{8} 2e^{-y^2} \sqrt{8-y^2} \, dy.$$ 

14. (10 points.) A lake is shaped like an upside down hemisphere, with radius 100 meters. The temperature of the water at depth $z$ meters below the surface is $20 - \frac{z}{10}$ degrees Centigrade. Set up an integral which represents the average temperature of the water in this lake. Include an explanation. Don’t try to evaluate the integral.

Solution: The volume of the lake is half the volume of a sphere with radius 100 meters. The volume is therefore $\frac{2}{3} \pi \cdot 100^3$ cubic meters.

Consider a thin layer of water at depth $z$ and with thickness $\Delta z$. It is shaped like a disk with radius $\sqrt{100^2 - z^2}$ and thickness $\Delta z$, so has a volume of

$$\pi \left( \sqrt{100^2 - z^2} \right)^2 \Delta z = \pi \left( 100^2 - z^2 \right) \Delta z,$$

and its temperature is $20 - \frac{z}{10}$ degrees Centigrade. Therefore the average temperature is

$$\frac{\int_0^{100} \left( 20 - \frac{z}{10} \right) \left( 100^2 - z^2 \right) \, dz}{\frac{2}{3} \pi \cdot 100^3}.$$ 

15. (15 points.) Find the centroid (center of mass) of the region between the curve $y = 4 - x^2$ and the $x$-axis.

Solution: By symmetry, the $x$-coordinate is 0. The $y$-coordinate is

$$\frac{\int_{-2}^{2} \frac{1}{2} [4 - x^2]^2 \, dx}{\int_{-2}^{2} [4 - x^2] \, dx}.$$ 

We have

$$\int_{-2}^{2} [4 - x^2] \, dx = \left( 4x - \frac{1}{3} x^3 \right) \bigg|_{-2}^{2} = 4 \cdot 2 - \frac{2^3}{3} - 4 \cdot (-2) + \frac{(-2)^3}{3} = \frac{32}{3}. $$
Also,
\[ \int_{-2}^{2} \frac{1}{2} [4 - x^2]^2 \, dx = \int_{-2}^{2} \left( 8 - 4x^2 + \frac{1}{2} x^4 \right) \, dx = \left. \left( 8x - \frac{4}{3} x^3 + \frac{1}{10} x^5 \right) \right|_{-2}^{2} = 8 \cdot 2 - \frac{4 \cdot 2^3}{3} + \frac{2^5}{10} - 8 \cdot (-2) - \frac{4 \cdot (-2)^3}{3} + \frac{(-2)^5}{10} = \frac{128}{15}. \]

Therefore the y-coordinate is
\[ \frac{\int_{-2}^{2} \frac{1}{2} [4 - x^2]^2 \, dx}{\int_{-2}^{2} [4 - x^2] \, dx} = \left( \frac{128}{15} \right) \left( \frac{32}{3} \right) = \frac{8}{5}. \]

So the center of mass is at \((0, \frac{8}{5})\).

16. (15 points.) Find the x-coordinate of the centroid (center of mass) of the region between the curve \( y = e^{-x} \), the y-axis, and the x-axis. (The region extends infinitely far to the right.)

Solution: The x-coordinate of the centroid is
\[ \frac{\int_{0}^{\infty} xe^{-x} \, dx}{\int_{0}^{\infty} e^{-x} \, dx}. \]

We evaluate the integrals. Both are improper. We have
\[ \int_{0}^{\infty} e^{-x} \, dx = -e^{-x} \bigg|_{0}^{\infty} = \lim_{x \to \infty} \left( -e^{-x} - (-e^{-0}) \right) = \lim_{x \to \infty} (1 - e^{-x}) = 1. \]

For the integral in the numerator, we first find the indefinite integral by integration by parts. We take \( u(x) = x, \) \( v'(x) = e^{-x}, \) \( u'(x) = 1, \) and \( v(x) = -e^{-x}. \) We get
\[ \int xe^{-x} \, dx = -xe^{-x} - \int (-e^{-x}) \, dx = xe^{-x} + \int e^{-x} \, dx = xe^{-x} - e^{-x} + C. \]

So
\[ \int_{0}^{\infty} xe^{-x} \, dx = (-xe^{-x} - e^{-x}) \bigg|_{0}^{\infty} = \lim_{x \to \infty} \left( -xe^{-x} - e^{-x} - (-0 \cdot e^{-0} - e^{-0}) \right) = \lim_{x \to \infty} \left( 1 - \frac{x}{e^x} - e^{-x} \right). \]

Now
\[ \lim_{x \to \infty} \frac{x}{e^x} = 1 \quad \text{and} \quad \lim_{x \to \infty} e^{-x} = 0. \]

Also, \( \lim_{x \to \infty} \frac{x}{e^x} \) has the indeterminate form \( \frac{\infty}{\infty} \), so we can evaluate it by L'Hospital’s Rule:
\[ \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0. \]

Putting things together, we get
\[ \int_{0}^{\infty} xe^{-x} \, dx = 1, \]
so the $x$-coordinate of the centroid is
\[ \frac{\int_0^\infty xe^{-x} \, dx}{\int_0^\infty e^{-x} \, dx} = \frac{1}{1} = 1. \]

17. (12 points.) Consider the region between the curve $y = x^3 + x$ and the lines $x = 0$ and $x = 2$, and the $x$-axis. It is rotated about the $x$-axis. Find the volume of the resulting solid.

Solution: (There is no picture in the file now. One may be provided later.) Here are four pictures. On the top right is a graph showing the lines and curves bounding the region to be rotated, and (dotted lines) the reflections of the curve in the $x$-axis (the axis of rotation). The top left and bottom right pictures are two views of an opaque version of the solid of revolution. The bottom right is a partially transparent view which also shows a disk shaped slice (at $x = \frac{3}{2}$).

We use vertical disks. For $0 \leq x \leq 2$, a cross section at $x$ and perpendicular to the $x$-axis is a disk with radius $x^3 + x$. So its area is
\[ \pi (x^3 + x)^2 = \pi (x^6 + 2x^4 + x^2). \]
If it has thickness $\Delta x$ for small $\Delta x$, its volume is $\pi (x^6 + 2x^4 + x^2) \Delta x$. This leads to the integral

$$\int_0^2 \pi (x^6 + 2x^4 + x^2) \, dx = \pi \left( \frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3} \right) \bigg|_0^2 = \pi \left( \frac{2^7}{7} + \frac{2^6}{5} + \frac{2^3}{3} \right).$$

18. (15 points.) Consider the region between the curve $y = e^x + x + 1$ and the lines $x = 0$ and $x = 2$, and the $x$-axis. It is rotated about the $y$-axis. Find the volume of the resulting solid.

**Solution (sketch):** (There is no picture in the file now. One may be provided later.) Here are four pictures. On the top right is a graph showing the lines and curve bounding the region to be rotated, and (dotted lines) the reflection of the boundary curves in the $y$-axis (the axis of rotation). The top left and bottom right show opaque and partially transparent views of the solid of revolution. The bottom right is a partially transparent view which also shows a cylindrical shell (at $x = 1$).
We use cylindrical shells with vertical axis. For $0 \leq x \leq 2$, the cylindrical shell at this value of $x$ and with thickness $\Delta x$ has radius $x$ and height $e^x + x + 1$, hence volume approximately $2\pi x(e^x + x + 1) \Delta x$. This leads to the integral

$$\int_0^2 2\pi x(e^x + x + 1) \, dx = 2\pi \int_0^2 xe^x \, dx + 2\pi \int_0^2 x \, dx + 2\pi \int_0^2 1 \, dx.$$ 

Now

$$2\pi \int_0^2 x^2 \, dx = \frac{2\pi x^3}{3} \bigg|_0^2 = \frac{16\pi}{3} \quad \text{and} \quad 2\pi \int_0^2 x \, dx = \pi x^2 \bigg|_0^2 = 4\pi.$$

We use integration by parts for the remaining integral, taking $u(x) = 2\pi x$, $v'(x) = e^x$, $u'(x) = 2\pi$, and $v(x) = e^x$. We get

$$2\pi \int_0^2 xe^x \, dx = 2\pi xe^x \bigg|_0^2 - \int_0^2 2\pi e^x \, dx = 2\pi xe^x \bigg|_0^2 - 2\pi e^x \bigg|_0^2$$

$$= 4\pi e^2 - 0 - (2\pi e^2 - 2\pi) = 2\pi e^2 + 2\pi.$$

Putting everything together gives the total volume

$$\frac{16\pi}{3} + 4\pi + 2\pi e^2 + 2\pi = \frac{34\pi}{3} + 2\pi e^2.$$

19. (12 points.) Consider the region between the curves $y = e^{-x}$ and $e^{-2x}$, and the lines $x = 1$ and $x = \ln(6)$. It is rotated about the $x$-axis. Find the volume of the resulting solid.
Solution: (There is no picture in the file now. One may be provided later...) Here are three pictures. On the right is a graph showing the lines and curves bounding the region to be rotated, and (dotted lines) the reflections of the curves in the $x$-axis (the axis of rotation). The center picture is an opaque solid of revolution. The right is a partially transparent view which also shows a washer shaped slice (at $x = \frac{3}{2}$).

We use vertical washers. For $1 \leq x \leq \ln(6)$, a cross section at $x$ and perpendicular to the $x$-axis is an annulus with inner radius $e^{-2x}$ and outer radius $e^{-x}$. So its area is $\pi(e^{-x})^2 - \pi(e^{-2x})^2$. If it has thickness $\Delta x$ (for small $\Delta x$), its volume is approximately $(\pi(e^{-x})^2 - \pi(e^{-2x})^2) \Delta x$. This leads to the integral

$$
\int_1^{\ln(6)} (\pi(e^{-x})^2 - \pi(e^{-2x})^2) \, dx = \int_1^{\ln(6)} (\pi(e^{-2x}) - \pi(e^{-4x})) \, dx
$$

$$
= \pi \left( -\frac{e^{-2x}}{2} + \frac{e^{-4x}}{4} \right) \bigg|_1^{\ln(6)}
$$

$$
= \pi \left( -\frac{1}{2 \cdot 6^2} + \frac{1}{4 \cdot 6^4} + \frac{1}{2e^2} - \frac{1}{4e^4} \right).
$$

20. (15 points.) A solid has a curved base which is the region between the curve $y = 4 - x^2$ and the $x$-axis. Cross sections perpendicular to the $x$-axis are isosceles triangles with height half the base length. Find the volume of this solid.

Solution: The graph of $y = 4 - x^2$ intersects the $x$-axis at $x = -2$ and $x = 2$. For $-2 \leq x \leq 2$, consider the cross section of the solid perpendicular to the $x$-axis. It has base length $4 - x^2$, so it has height $\frac{1}{2}(4 - x^2)$. Thus it has area

$$
\frac{1}{2} \left( (4 - x^2) \cdot \frac{1}{2}(4 - x^2) \right) = \frac{1}{4}(4 - x^2)^2.
$$

If a cross section at this position has thickness $\Delta x$, and $\Delta x$ is small, then it has volume approximately $\frac{1}{4}(4 - x^2)^2 \Delta x$.

Now consider a subdivision of $[-2, 2]$ into $n$ intervals, each of length $\Delta x = 4/n$. (We don’t have to use equal length intervals; this is just for convenience.) For $k = 1, 2, \ldots, n$ let $x_k$ be a
point in the $k$-th interval. Then the volume is approximately

$$
\sum_{k=1}^{n} \frac{1}{4} (4 - x_k^2)^2 \Delta x.
$$

This looks like a Riemann sum for

$$
\int_{-2}^{2} \frac{1}{4} (4 - x^2)^2 \, dx.
$$

So the volume is

$$
\int_{-2}^{2} \frac{1}{4} (4 - x^2)^2 \, dx = \int_{-2}^{2} \frac{1}{4} (16 - 8x^2 + x^4) \, dx = \left( 4x - \frac{2x^3}{3} + \frac{x^5}{20} \right) \bigg|_{-2}^{2} = \frac{128}{15}.
$$

(To get full credit, a solution does not have to contain an explicit Riemann sum as in (4) above. But the reasoning behind the expression

$$
\sum_{k=1}^{n} \frac{1}{4} (4 - x_k^2)^2 \Delta x \quad \text{or} \quad \int_{-2}^{2} \frac{1}{4} (4 - x^2)^2 \, dx
$$

must be made clear.)

21. (12 points.) Consider the region between the curve $y = \sqrt{2x}$ and the lines $x = 8$ and $y = 2$. It is rotated about the $x$-axis. Find the volume of the resulting solid.

Solution: (There is no picture in the file now. One may be provided later... ) Here are three pictures. On the right is a graph showing the lines and curve bounding the region to be rotated, and (dotted lines) the reflections of the curve and horizontal line in the $x$-axis (the axis of rotation). The other two pictures are two views of the resulting solid of revolution (opaque), the first seen from smaller values of $x$ and the second seen from larger values of $x$.

Here are three pictures related to methods for finding the volume, all with semitransparent versions of the solid. The left and middle pictures are two views of a washer shaped slice (at $x = 6$). The right picture shows a cylindrical shell (at $y = 3$).
We use vertical washers, which give an integral with respect to $x$. One limit of integration is $x = 8$. The other limit of integration is $x$-coordinate of the point at which the line $y = 2$ intersects the curve $y = \sqrt{2x}$. So we solve $2 = \sqrt{2x}$, which has the unique solution $x = 2$.

For $2 \leq x \leq 8$, a cross section at $x$ and perpendicular to the $x$-axis is an annulus with inner radius 2 and outer radius $\sqrt{2x}$. So its area is
\[ \pi (\sqrt{2x})^2 - \pi (2)^2 = 2\pi x - 4\pi. \]
If it has thickness $\Delta x$ for small $\Delta x$, its volume is approximately $(2\pi x - 4\pi) \Delta x$. This leads to the integral
\[ \int_2^8 (\pi (\sqrt{2x})^2 - \pi (2)^2) \, dx = \int_2^8 (2\pi x - 4\pi) \, dx \]
\[ = (\pi x^2 - 4\pi x) \bigg|_2^8 = (64\pi - 32\pi) - (4\pi - 8\pi) = 36\pi. \]

Alternate solution (sketch): We use cylindrical shells with horizontal axis, which give an integral with respect to $y$. One limit of integration is $y = 2$. The other is the $y$-coordinate of the point at which the line $x = 8$ intersects the curve $y = \sqrt{2x}$. So we find $y = \sqrt{2 \cdot 8} = 4$. We also need to rewrite the equation $y = \sqrt{2x}$ as $x = \frac{1}{2}y^2$.

Consider a height $y$ with $2 \leq y \leq 4$. The cylindrical shell at this value of $y$ has radius $y$ and length $8 - \frac{1}{2}y^2$ (the difference of values of $x$). So its area is $2\pi y(8 - \frac{1}{2}y^2)$. If it has thickness $\Delta y$, then it has volume approximately $2\pi y(8 - \frac{1}{2}y^2) \Delta y$. This leads to the integral
\[ \int_2^4 2\pi y \left(8 - \frac{1}{2}y^2\right) \, dy = \int_2^4 2\pi y(16 - y^2) \, dy. \]

22. (15 points.) Consider the region between the curve $y = e^x + x$ and the lines $x = 0$ and $x = 2$, and the $x$-axis. It is rotated about the $x$-axis. Find the volume of the resulting solid.
Solution (sketch): (There is no picture in the file now. One may be provided later.) Here are four pictures. On the top right is a graph showing the lines and curves bounding the region to be rotated, and (dotted lines) the reflections of the curve in the x-axis (the axis of rotation). The top left and bottom right pictures are two views of an opaque version of the solid of revolution. The bottom right is a partially transparent view which also shows a disk shaped slice (at $x = \frac{3}{2}$).

We use vertical disks, which give an integral with respect to $x$. For $0 \leq x \leq 2$, a cross section at $x$ and perpendicular to the $x$-axis is a disk with radius $e^x + x$. So its area is

$$\pi(e^x + x)^2 = \pi e^{2x} + 2\pi xe^x + \pi x^2.$$ 

If it has thickness $\Delta x$ for small $\Delta x$, its volume is $(\pi e^{2x} + 2\pi xe^x + \pi x^2) \Delta x$. This leads to the integral

$$\int_0^2 (\pi e^{2x} + 2\pi xe^x + \pi x^2) \, dx = \int_0^2 \pi e^{2x} \, dx + \int_0^2 2\pi xe^x \, dx + \int_0^2 \pi x^2 \, dx.$$ 

Now

$$\int_0^2 \pi e^{2x} \, dx = \pi \left( \frac{e^{2x}}{2} \right) \bigg|_0^2 = \frac{\pi e^4}{2} - \frac{\pi}{2}$$

and

$$\int_0^2 \pi x^2 \, dx = \pi \left( \frac{x^3}{3} \right) \bigg|_0^2 = \frac{8\pi}{3}.$$
We use integration by parts for the remaining integral, taking \( u(x) = 2\pi x \), \( v'(x) = e^x \), \( u'(x) = 2\pi \), and \( v(x) = e^x \). We get

\[
\int_0^2 2\pi x e^x \, dx = \left[ 2\pi x e^x \right]_0^2 - \int_0^2 2\pi e^x \, dx = 2\pi x e^x \bigg|_0^2 - 2\pi e^x \bigg|_0^2
\]

\[
= 4\pi e^2 - 0 - (2\pi e^2 - 2\pi) = 2\pi e^2 + 2\pi.
\]

Putting everything together gives the total volume

\[
\frac{\pi e^4}{2} - \frac{\pi}{2} + \frac{8\pi}{3} + 2\pi e^2 + 2\pi = \pi \left( \frac{25}{6} + \frac{e^4}{2} + 2e^2 \right).
\]

23. (10 points.) Let \( a \) be a constant. Find \( \int \cos^2(at) \, dt \).

Solution: We use the formula \( \cos^2(x) = \frac{\cos(2x) + 1}{2} \) with \( x = at \). Thus

\[
\int \cos^2(at) \, dt = \int \frac{\cos(2at) + 1}{2} \, dt = \frac{\sin(2at)}{4a} + t + C.
\]

Alternate solution (outline): This problem can also be done by integrating by parts, using \( \sin^2(at) + \cos^2(at) = 1 \), and then solving for the original integral.

24. (12 points.) Find \( \int \cos^2(5x) \sin^3(5x) \, dx \).

Solution: We write

\[
\int \cos^2(5x) \sin^3(5x) \, dx = \int \cos^2(5x) (1 - \cos^2(5x)) \sin(5x) \, dx = \int (\cos^2(5x) - \cos^4(5x)) \sin(5x) \, dx.
\]

Use the substitution \( w = \cos(5x) \). Then \( dw = -5\sin(5x) \, dx \). So \( \sin(5x) \, dx = -\frac{1}{5} \, dw \). Therefore

\[
\int (\cos^2(5x) - \cos^4(5x)) \sin(5x) \, dx = -\frac{1}{5} \int w^2 (1 - w^2) \, dw = -\frac{1}{5} \left( \frac{w^3}{3} - \frac{w^5}{5} \right) + C = -\frac{\cos^3(5x)}{15} + \frac{\cos^5(5x)}{25} + C.
\]

25. (15 points.) Find \( \int \tan^5(3z) \, dz \).
Solution: We have
\[ \int \tan^5(3z) \, dz = \int (\sec^2(3z) - 1) \tan^3(3z) \, dz = \int \sec^2(3z) \tan^3(3z) \, dz - \int \tan^3(3z) \, dz \]
\[ = \int \sec^2(3z) \tan^3(3z) \, dz - \int (\sec^2(3z) - 1) \tan(3z) \, dz \]
\[ = \int \sec^2(3z) \tan^3(3z) \, dz - \int \sec^2(3z) \tan(3z) \, dz + \int \tan(3z) \, dz. \]

Using the substitution \( w = \tan(3z) \), so \( dw = 3 \sec^2(3z) \, dz \), on each of the first two integrals, we get
\[ \int \sec^2(3z) \tan^3(3z) \, dz = \int \frac{w^3}{3} \, dw = \frac{\tan^4(3z)}{12} + C_1. \]

and
\[ \int \sec^2(3z) \tan(3z) \, dz = \int \frac{w}{3} \, dw = \frac{\tan^2(3z)}{6} + C_2. \]

On the last integral, we use the substitution \( u = \cos(3z) \), so \( du = -3 \sin(3z) \, dz \), getting
\[ \int \tan(3z) \, dz = \int \frac{\sin(3z)}{\cos(3z)} \, dz = \int \left( -\frac{1}{3u} \right) \, du = -\frac{1}{3} \ln(|u|) + C_3 = -\frac{\ln(|\cos(3z)|)}{3} + C_3. \]

Putting everything together, and combining the constants of integration,
\[ \int \sec^2(3z) \tan^3(3z) \, dz = \frac{\tan^4(3z)}{12} - \frac{\tan^2(3z)}{6} - \frac{\ln(|\cos(3z)|)}{3} + C. \]

Alternate solution (sketch): We have
\[ \int \tan^5(3z) \, dz = \int (\sec^2(3z) - 1)^2 \tan(3z) \, dz \]
\[ = \int \sec^4(3z) \tan(3z) \, dz - \int 2 \sec^2(3z) \tan(3z) \, dz + \int \tan(3z) \, dz. \]

The last integral is
\[ \int \tan(3z) \, dz = -\frac{\ln(|\cos(3z)|)}{3} + C_3, \]
as in the first solution. On the other two integrals, use the substitution \( v = \sec(3z) \), so \( dv = 3 \sec(3z) \tan(3z) \, dz \). to turn them into
\[ \frac{1}{3} \int v^3 \, dv = \frac{1}{12} \sec^4(3z) + C_4 \quad \text{and} \quad \frac{1}{3} \int v \, dv = \frac{1}{6} \sec^2(3z) + C_5. \]

26. (12 points.) Find \( \int_{6}^{11} \frac{10}{(t-2)(t+3)} \, dt \).

Solution: We use partial fractions. We write
\[ \frac{10}{(t-2)(t+3)} = \frac{a}{t-2} + \frac{b}{t+3}. \]
and solve for $a$ and $b$. For all $t \neq 2, 3$ we have
\[
\frac{10}{(t-2)(t+3)} = \frac{a}{t-2} + \frac{b}{t+3} = \frac{a(t+3)}{(t-2)(t+3)} + \frac{b(t-2)}{(t-2)(t+3)} = \frac{(a+b)t + (3a-2b)}{(t-2)(t+3)}.
\]
This can only happen if $a + b = 0$ and $3a - 2b = 10$. These two equations have the simultaneous solution $a = 2$ and $b = -2$. Thus
\[
\frac{10}{(t-2)(t+3)} = \frac{2}{t-2} - \frac{2}{t+3},
\]
and
\[
\int_6^{11} \frac{10}{(t-2)(t+3)} dt = \int_6^{11} \frac{2}{t-2} dt - \int_6^{11} \frac{2}{t+3} dt = 2 \ln(|t-2|) \bigg|_6^{11} - 2 \ln(|t+3|) \bigg|_6^{11} = 2 \ln(9) - 2 \ln(4) - 2 \ln(14) + 2 \ln(9) = 4 \ln(9) - 2 \ln(4) - 2 \ln(14).
\]

27. (12 points) Find $\int \frac{1}{(9 + x^2)^{5/2}} dx$.

**Solution:** Use the substitution $x = 3 \tan(w)$. Thus, $dx = 3 \sec^2(w) dw$, so
\[
\int \frac{1}{(9 + x^2)^{5/2}} dx = \int \frac{1}{(9 + 9 \tan^2(w))^{5/2}} \cdot 3 \sec^2(w) dw = \frac{1}{81} \int \cos^3(w) dw
\]
\[
= \frac{1}{81} \int (1 - \sin^2(w)) \cos(w) dw
\]
\[
= \frac{1}{81} \int \cos(w) dw - \frac{1}{81} \int \sin^2(w) \cos(w) dw
\]
\[
= \frac{\sin(w)}{81} - \frac{\sin^3(w)}{243} + C.
\]
Now
\[
\sin^2(w) = \frac{\sin^2(w)}{\cos^2(w)} \cdot \frac{1}{\sec^2(w)} = \frac{\tan^2(w)}{1 + \tan^2(w)} = \frac{\frac{1}{9} x^2}{1 + \frac{1}{9} x^2} = \frac{x^2}{9 + x^2}.
\]
Since $\sin(w)$ and $x$ are either both positive or both negative, we get
\[
\sin(w) = \frac{x}{(x^2 + 9)^{1/2}}.
\]
Therefore
\[
\int \frac{1}{(9 + x^2)^{5/2}} dx = \frac{x}{81(x^2 + 9)^{1/2}} - \frac{x^3}{243(x^2 + 9)^{3/2}} + C.
\]
(The last steps are necessary.)

28. (12 points) Find $\int x^2 \sec^4(2x^3) \tan^2(2x^3) dx$. 
Solution: Use the substitution \( w = \tan(2x^3) \). Then \( dw = 6x^2 \sec^2(2x^3) \, dx \). Therefore
\[
\int x^2 \sec^4(2x^3) \tan^2(2x^3) \, dx = \int x^2 \sec^2(2x^3) \tan^2(2x^3) \, dx
\]
\[
= \frac{1}{6} \int (1 + w^2)w^2 \, dw = \frac{w^3}{18} + \frac{w^5}{30} + C
\]
\[
= \frac{\tan^3(2x^3)}{18} + \frac{\tan^5(2x^3)}{30} + C.
\]

29. (12 points.) Find \( \int \frac{1}{(1 - 4x^2)^{3/2}} \, dx \).

Solution: Use the substitution \( 2x = \sin(w) \). Then \( dx = \frac{\cos(w)}{2} \, dw \). So
\[
\int \frac{1}{(1 - 4x^2)^{3/2}} \, dx = \int \frac{1}{(1 - \sin^2(w))^{3/2}} \cdot \frac{\cos(w)}{2} \, dw = \int \frac{1}{2 \cos^2(w)} \, dw
\]
\[
= \frac{1}{2} \int \sec^2(w) \, dw = \frac{\tan(w)}{2} + C.
\]
Now
\[
\tan^2(w) = \frac{\sin^2(w)}{\cos^2(w)} = \frac{\sin^2(w)}{1 - \sin^2(w)} = \frac{4x}{1 - 4x^2},
\]
and \( x \) and \( \tan(w) \) are either both positive or both negative, so
\[
\tan(w) = \frac{2x}{\sqrt{1 - 4x^2}}
\]
Therefore
\[
\int \frac{1}{(1 - 4x^2)^{3/2}} \, dx = \frac{x}{\sqrt{1 - 4x^2}} + C.
\]

30. (8 points.) A nonlinear industrial size spring has restoring force \( 2x + \frac{1}{5}x^3 \) Newtons when stretched \( x \) meters beyond its natural (equilibrium) length. Find the work done when stretching it from 1 meter beyond its natural length to 2 meters beyond its natural length.

Solution: Remembering that work is force times distance, we need to calculate
\[
\int_1^2 \left(2x + \frac{1}{5}x^3\right) \, dx = \left[x^2 + \frac{x^4}{20}\right]_1^2 = \left(2^2 + \frac{2^4}{20}\right) - \left(1^2 + \frac{1^4}{20}\right) = \frac{15}{4}.
\]
So the work is \( 15/4 \) Newton-meters.
(The units in the final answer are required.)

31. (12 points.) Find \( \int \frac{2t^2 + t}{1 + t^2} \, dt \).
Solution: We write
\[
\frac{2t^2 + t}{1 + t^2} = \frac{2(1 + t^2) + t - 2}{1 + t^2} = 2 + \frac{t - 2}{1 + t^2} = 2 + \frac{t}{1 + t^2} - \frac{2}{1 + t^2}.
\]
So
\[
\int \frac{2t^2 + t}{1 + t^2} \, dt = \int 2 \, dt + \int \frac{t}{1 + t^2} \, dt - \int \frac{2}{1 + t^2} \, dt = 2t + \frac{\ln(1 + t^2)}{2} - 2 \arctan(t) + C.
\]

32. (12 points.) A cable on the third moon of the planet Groggxth is 2 meters long and has linear density \(\frac{16}{1 + x^2}\) kilograms per meter at the point \(x\) meters from the heavy end. It is resting on the floor of a flat room. How much work is needed to hang it from a hook on the ceiling, with the heavy end at the top? The ceiling of the room is 3 meters high, and the acceleration due to gravity on the surface of this moon is \(2 \text{ m/sec}^2\).

Solution: Take \(x\) in [0, 2), take \(\Delta x\) very small, and consider the section of cable which is between \(x\) and \(x + \Delta x\) meters from the heavy end. It has length \(\Delta x\) meters and linear density approximately \(\frac{16}{1 + x^2} \Delta x\) kilograms per meter, so it has mass approximately \(\frac{16}{1 + x^2} \Delta x\) kilograms. It must be lifted \(3 - x\) meters against a downwards acceleration of \(2 \text{ m/sec}^2\), so the work is
\[
2(3 - x) \cdot \frac{16}{1 + x^2} \Delta x
\]
Newton-meters. Therefore the total work is
\[
\int_0^2 2(3 - x) \cdot \frac{16}{1 + x^2} \, dx = \int_0^2 \frac{96}{1 + x^2} \, dx - \int_0^2 \frac{32x}{1 + x^2} \, dx
\]
\[
= 96 \arctan(x) \bigg|_0^2 - 16 \ln(1 + x^2) \bigg|_0^2 
\]
\[
= 96 \arctan(2) - 16 \ln(5).
\]
So the work is \(96 \arctan(2) - 16 \ln(5)\) Newton-meters.

(The units in the final answer, and the simplifications \(\arctan(0) = 0\) and \(\ln(1) = 0\), are required.)

33. (12 points.) You are building a stone pyramid on the planet Yuggxth. The stone you use is flat on the ground, and has a density of 5000 kilograms per cubic meter. The pyramid has a square base, which is 20 meters on each side, and is 10 meters high; its sides are straight. The acceleration due to gravity on the surface of Yuggxth is \(10 \text{ m/sec}^2\). How much work is needed to construct the pyramid?

Solution: Consider a height \(h\) meters, with \(0 \leq h \leq 10\). The cross section of the pyramid at height \(h\) is a square whose side length is \(20 - 2h\) meters. (It must be linear in \(h\), most be 20 when \(h = 0\), and must be 0 when \(h = 10\).) Therefore it has area \((20 - 2h)^2\). A later of rock at this height with small thickness \(\Delta h\) has volume \((20 - 2h)^2 \Delta h\) cubic meters, so it has mass
5000(20 − 2h)^2 \Delta h \text{ kilograms. It must be lifted } h \text{ meters against a downwards acceleration of } 10 \text{ m/sec}^2, \text{ so the work is } 10h \cdot 5000(20 − 2h)^2 \Delta h \text{ Newton-meters. Therefore the total work is}

\[ \int_0^{10} 10h \cdot 5000(20 − 2h)^2 \, dh = 50000 \int_0^{10} (400h − 80h^2 + 4h^3) \, dh \]

\[ = 50000 \left( 200h^2 - \frac{80h^3}{3} + h^4 \right) \bigg|_0^{10} = \frac{5 \cdot 10^8}{3}. \]

So the work is \(5 \cdot 10^8/3\) Newton-meters.

(The units in the final answer are required.)

34. (12 points.) The height in meters of a hot air balloon at time \(t\) hours after noon on 3 January is modelled by

\[ h(t) = 1000 + 20t - 10 \sin \left( \frac{\pi t}{12} \right). \]

According to the model, what is average height of the hot air balloon between 2:00 pm and 5:00 pm on 3 January?

Solution: We need to find the average value of \( h(t) \) for \( t \) in \([2, 5]\), which is

\[ \frac{1}{5 - 2} \int_2^5 h(t) \, dt = \frac{1}{3} \int_2^5 \left( 1000 + 20t - 10 \sin \left( \frac{\pi t}{12} \right) \right) \, dt \]

\[ = \frac{1}{3} \left( 1000t + 10t^2 + \frac{120}{\pi} \cos \left( \frac{\pi t}{12} \right) \right) \bigg|_2^5 \]

\[ = \frac{1}{3} \left( 1000 \cdot 5 + 10 \cdot 5^2 + \frac{120}{\pi} \cos \left( \frac{5\pi}{12} \right) \right) \]

\[ - \frac{1}{3} \left( 1000 \cdot 2 + 10 \cdot 2^2 + \frac{120}{\pi} \cos \left( \frac{2\pi}{12} \right) \right) \]

\[ = \frac{1}{3} \left( 3000 + 10(25 - 4) + \frac{120}{\pi} \left( \cos \left( \frac{5\pi}{12} \right) - \cos \left( \frac{\pi}{6} \right) \right) \right) \]

\[ = 1070 + \frac{40}{\pi} \left( \cos \left( \frac{5\pi}{12} \right) - \frac{\sqrt{3}}{2} \right). \]

So the average height is \(1070 + \frac{40}{\pi} \left( \cos \left( \frac{5\pi}{12} \right) - \sqrt{3}/2 \right)\) meters.

(The units in the final answer are required.)

35. (15 points.) A lake on the planet Yuggxth is held in place by a dam which is shaped like an upside down semicircle with radius 7 meters. The lake is filled with a poisonous liquid (which, however, is very nutritious to the fire breathing monsters which live there) with a density of 500 kilograms per cubic meter. The acceleration due to gravity on the surface of Yuggxth is \(4 \text{ m/sec}^2\). What force does the lake exert on the dam?

Solution: Consider the depth \( h \) meters. Above each square meter, there are \( h \) cubic meters of liquid, with a mass of \( 500 \cdot h \) kilograms. Therefore the downwards force is \( 4 \cdot 500 \cdot h = 2000h \)
Thus the pressure at depth $h$ meters is $2000h$ Newtons per square meter. A horizontal strip of the dam at this depth is $2\sqrt{49 - h^2}$ meters long. Therefore the total force is

$$\int_0^7 2000h \cdot 2\sqrt{49 - h^2} \, dh = 4000 \int_0^7 h\sqrt{49 - h^2} \, dh$$

$$= \frac{4000}{3} (49 - h^2)^{3/2} \bigg|_0^7 = \frac{7^3 \cdot 4000}{3}.$$ 

So the work is $7^3 \cdot 4000/3$ Newton-meters.

(The units in the final answer are required.)