Math 252  Midterm 1 Solutions

1. (8 points) State the Fundamental Theorem of Calculus (either of the versions I gave in class).

   Solution: Let \( f \) be continuous on \([a, b]\), and let \( F \) be a function on \([a, b]\) with \( F' = f \). Then \( \int_a^b f(t) \, dt = F(b) - F(a) \).

   OR

   Let \( f \) be continuous on \([a, b]\), and define \( F(x) = \int_a^x f(t) \, dt \). Then \( F'(x) = f(x) \).

   You must say in your answer what the connection between \( F \) (if you use that letter) and \( f \) is.

2. (6 points) Use the left hand Riemann sum with 3 subintervals to estimate \( \int_{-1}^1 \sin^2(x) \, dx \). Show your work by writing out all of the terms of the sum.

   Solution:

   \[
   \frac{2}{3} \sin^2(-1) + \frac{2}{3} \sin^2\left(-\frac{1}{3}\right) + \frac{2}{3} \sin^2\left(\frac{1}{3}\right) \approx 0.614791.
   \]

3. (8 points/part) Find all antiderivatives of the following functions:
   (a) \( f(x) = 3x^2 \). (Also, find an antiderivative \( F \) such that \( F(-1) = 0 \).)

   Solution: The most general antiderivative is \( F(x) = x^3 + C \). To get \( F(-1) = 0 \), use \( F(x) = x^3 + 1 \).

   (b) \( g(t) = t\sqrt{t} \).

   Solution: \( g(t) = t^{3/2} \), so the most general antiderivative is \( G(t) = \frac{2}{5} t^{5/2} + C \).

   (c) \( h(z) = -ze^{-z^2} \).

   Solution: The most general antiderivative is \( H(z) = \frac{1}{2} e^{-z^2} + C \).

   (d) \( q(x) = x \sin(3x^4) \), in terms of elementary functions and a function \( F \) such that \( F'(x) = \sin(3x^2) \).

   Solution: The most general antiderivative is \( Q(x) = \frac{1}{2} F(x^2) + C \).
4. (6 points) At the right is a graph of the function $f(x) = e^{-x^2}$. Using its symmetry about the $y$-axis, and the information $\int_0^1 e^{-x^2} \, dx = 0.746824$ and $\int_1^2 e^{-x^2} \, dx = 0.135257$ (correct to 6 decimal places), find $\int_{-1}^2 e^{-x^2} \, dx$. Show your steps.

Solution:

\[
\int_{-1}^2 e^{-x^2} \, dx = 2 \int_0^1 e^{-x^2} \, dx + \int_1^2 e^{-x^2} \, dx \approx 1.628905.
\]

Note that $\int_{-1}^0 e^{-x^2} \, dx = \int_0^1 e^{-x^2} \, dx$ by symmetry. Mathematica gives 1.628906; the discrepancy is due to accumulation of rounding errors.

5. Consider, but do not solve, the following problem: We wish to estimate $\int_1^5 e^{-x^2} \, dx$ to an accuracy of 0.05.

(a) (6 points) What do we need to check to know that the standard error estimate for Riemann sums applies to this problem?

\textbf{Solution:} You need to check that $f(x) = e^{-x^2}$ is either nondecreasing or nonincreasing on $[1, 5]$. Its graph appears in the previous problem, and it is clearly nonincreasing.

(b) (10 points) How many subdivisions of the interval should be used to ensure that the right hand Riemann sum for the integral satisfies the required accuracy? (Do not compute the Riemann sum).

\textbf{Solution:} Let $n$ be the number of subintervals. It is necessary to have

\[
\Delta x(f(1) - f(5)) = \left( \frac{5 - 1}{n} \right) (f(1) - f(5)) \leq 0.05.
\]

This means

\[
n \geq \frac{4(f(1) - f(5))}{0.05} \approx 29.4304.
\]

Since $n$ is an integer, take $n = 30$.

6. At the right is the graph of the function $y = f(x)$. (The lines are straight.)

(a) (18 points) On the grid below, sketch the graph of the antiderivative $F(x)$ of $f(x)$ which satisfies $F(0) = 7$. Label all local maximums, local minimums, and inflection points of $F$, giving their $x$ and $y$ coordinates.
Solution: The shape of the graph follows from the coordinates requested, plus the points at $x = 0$ and $x = 6$. I can’t label them easily on the graph, but here are the calculations. There is a local maximum at $x = 1$, where the derivative $f$ of $F$ changes from positive to negative, and similarly a local minimum at $x = 5$. There is an inflection point at $x = 3$, where the derivative $f$ of $F$ has a local minimum. The areas of the two triangular regions above the $x$-axis in the graph of $f$ are each 1, and each half of the area of the lower triangular region has area 4. So $F(0) = 7$, $F(1) = 7+1 = 8$, $F(3) = 8-4 = 4$, $F(5) = 4-4 = 0$, and $F(6) = 0 + 1 = 1$.

(b) (8 points) Find the average value of $f$ on the interval $[0,6]$.

Solution: By adding up the areas, or by considering $F(6) - F(1)$ from the previous part, one finds that $\int_0^6 f(t) \, dt = -6$. Divide by the length of the interval to see that the average value is $-1$.

7. (6 points) If $f(t)$ is measured in meters/second$^2$ and $t$ is measured in seconds, what are the units of $\int_a^b f(t) \, dt$?

Solution: meters/second$^2 \cdot$ seconds = meters/second.

Extra credit. (This problem will only be counted if you get a grade of B or better on the main part of this exam.)

When the brakes of a car are applied hard, the result is a fixed constant deceleration until the car comes to a stop. Use antiderivatives to show that if the car is going twice as fast when the brakes are applied, it travels four times as far before it stops.

Note: Many people spent time on this problem when they should have been doing the regular problems. Read the directions, and don’t try the extra credit until you are sure everything else is right.

Solution: Let $a$ be the constant deceleration (chosen positive), and let $v_0$ be the initial velocity the first time. Let $f(t)$ be the position $t$ units of time after applying the brakes at velocity $v_0$, and let $g(t)$ be the same thing for initial velocity $2v_0$.

Antidifferentiating twice gives $f(t) = -at^2/2 + c_1 t + c_2$ for constants $c_1$ and $c_2$. Since $f(0) = 0$, we get $c_2 = 0$. Since $f'(0) = v_0$, we get $c_1 = v_0$. So $f(t) = -at^2/2 + v_0 t$. The car stops when $f'(t) = 0$, and solving the
equation shows this happens at $t = v_0/a$. The distance travelled is then $f(v_0/a) = v_0^2/(2a)$.

Calculating $g$ the same way, we get $g(t) = -at^2/2 + 2v_0t$, $g'(t) = 0$ when $t = 2v_0/a$ (the time to stop is twice as long), and $g(2v_0/a) = 2v_0^2/a$, which is four times as far.

You can also notice that the distance is proportional to $v_0^2$ in the result for $f$, so doubling $v_0$ quadruples the distance.