Submit with your exam the statement (presumably handwritten), with your signature, “I understand the instructions on academic integrity for this exam, and will follow them. I understand that being found responsible for academic misconduct is considered cheating and will result in a score of zero on this exam.”

GENERAL INSTRUCTIONS (The problems are on the next page.)

1. The point values are as indicated; total 200 points (not counting extra credit).
2. This exam is open book and open notes. All work must be your own; you may not collaborate with others. This means that no interactive help (in person, internet based, or any other method) is allowed. A graphing calculator (not on your computer) is allowed, but computer algebra systems are not. All electronic devices other than your computer, cell phone (if needed for scanning), and calculator must be off and out of reach; the cell phone must be visible but off when not being used for scanning. Normal rules on academic integrity apply.
3. Starting time 19 March 2021 at 10:10 am Pacific Daylight Time. Your work must be submitted by 12:25 pm Pacific Daylight Time. For a nominally two hour exam, this allows 5 minutes for downloading at the beginning and 10 minute for scanning etc. at the end. You must stay until 12:15 am, even if you are done. Use the time to check answers, or prepare for Math 253.
4. Show enough of your work that your method is obvious. Be sure that every statement you write is correct and uses correct notation. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
5. Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
6. When exact values are specified, give answers such as $\frac{1}{7}$, $\sqrt{2}$, $\ln(2)$, or $\frac{2\pi}{9}$. Calculator approximations will not be accepted.
7. Final answers must always be simplified unless otherwise specified.

EXCERPTS FROM THE INSTRUCTIONS FOR REMOTE TESTING (See the separate sheet for the full set of instructions.)

1. You must be present in the normal class Zoom meeting (ID 916 3410 5223) while taking the exam. Zoom settings:
   • Virtual background off.
   • Video and web camera on, and set to show you and your workspace.
   • Microphone on.
   • Speaker on (to hear announcements).
2. Submit questions to the instructor via chat. They will be answered via chat, unless they are something the whole class should know about, in which case they will be answered via a spoken announcement.
3. Alternate submission, in case you have trouble with Canvas: by email to ncp@uoregon.edu. The file must be contained in the email message; files stored on Google Docs or similar sites will not be retrieved. (I will check the time stamps on email.)

Date: 19 March 2021.
1. (1 point) True or false: Final exams on Friday are #[&#%&(?#) [expletive deleted]]!!

Solution: It depends on how far behind you were in the course.

2. (12 points.) A fish hatchery puts trout in a lake at a constant rate of 3000 trout per year. Fishermen catch the trout at an instantaneous rate of $1/13$ of the number of trout per year. Unfortunately, this lake contains no locations suitable for spawning, so the trout don’t reproduce.

Set up a differential equation which models the population of trout in this lake as a function of time. Be sure to state what your variables mean. Do not try to solve the equation.

Solution: Let $F(t)$ be the number of trout in the lake at time $t$, with $t$ measured in years. Here are two contributions to the rate of change of $F(t)$: a positive contribution from the hatchery, which is 3000, and a negative contribution from fishing, which is $- (1/13) F(t)$. So the equation is

$$F'(t) = 3000 - (1/13) F(t).$$

3. (12 points.) Here is a direction field:

On this graph, draw the graphs of the five solutions $y(x)$ to the equations satisfying the initial conditions $y(0) = -2, -1, 0, 1, 2$.

(If you are unable to print this page, draw the graphs on a blank page, with axes labelled and scales marked.)

Solution: Here are the requested solution curves.
In case anyone is interested, the differential equation used to produce this direction field is $y'(x) = x^2/(y^2 + 4)$.

4. (15 points.) Let $q$ be a continuous function defined for all real numbers, and let $Q$ be a function such that $Q'(x) = q(x)$ for all real $x$. Find $\int \sin(x)q(\cos(x)) \, dx$. Your answer may involve $q$, $Q$, or both.

Solution: We use the substitution $u = \cos(x)$, so $du = -\sin(x) \, dx$. This gives

$$\int \sin(x)q(\cos(x)) \, dx = \int (-q(u)) \, du = -Q(u) + C = -Q(\cos(x)) + C.$$ 

5. (15 points.) Consider the differential equation

$$y'(t) = [y(t)]^2 - \frac{2}{t^2},$$

which is to hold for all $t > 0$. Determine whether the function $y(t) = -\frac{2}{t}$ for $t > 0$ satisfies this equation. Be sure to show your reasoning carefully and using correct notation. (No credit for just “yes” or just “no”, even if it happens to be right.)

Solution: Write $y(t) = -2t^{-1}$. So $y'(t) = 2t^{-2}$. Also,

$$[y(t)]^2 - \frac{2}{t^2} = \left(-\frac{2}{t}\right)^2 - \frac{2}{t^2} = \frac{4}{t^2} - \frac{2}{t^2} = \frac{2}{t^2},$$
4 SOLUTIONS TO FINAL EXAM

which is indeed equal to \( y'(t) \). So \( y(t) = -\frac{2}{t} \) satisfies the differential equation.

6. (12 points.) Use a suitable substitution to convert the integral \( \int (9 + x^2)^{1/2} \, dx \) into one involving only integer powers of the standard trigonometric functions. Do not evaluate the resulting integral.

**Solution:** Since \( \tan^2(\theta) + 1 = \sec^2(\theta) \), we take \( x = 3\tan(\theta) \), with \( \theta \) in \( \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \), so \( dx = 3\sec^2(\theta) \, d\theta \). Thus (using \( \sec(\theta) > 0 \) when we take the square root)

\[
\int (9 + x^2)^{1/2} \, dx = \int 3\sec^2(\theta) (9 + 9\tan^2(\theta))^{1/2} \, d\theta \\
= \int 3\sec^2(\theta) (9\sec^2(\theta))^{1/2} \, d\theta = \int 3\sec^2(\theta) \cdot 3\sec(\theta) \, d\theta = 9 \int \sec^3(\theta) \, d\theta.
\]

7. (15 points.) Find the area of the region bounded by the curves \( x = y^2 - 4y \) and \( x = 2y - y^2 \). (It may help to draw a picture of the region first.)

**Solution:** Here is a picture of the region:

![Graph](image)

We use horizontal slices. The \( y \)-coordinates of the two points where the curves intersect are found by solving the equation

\[
y^2 - 4y = 2y - y^2,
\]
which simplifies to \( 2y^2 - 6y = 0 \). Factoring, we get \( 2y(y - 3) = 0 \), which has the solutions \( y = 0 \) and \( y = 3 \). For \( 0 < y < 3 \), the second curve is to the right of the first, so the area is

\[
\int_0^3 \left[ (2y - y^2) - (y^2 - 4y) \right] \, dy = \int_0^3 (6y - 2y^2) \, dy \\
= \left( 3y^2 - \frac{2}{3}y^3 \right) \bigg|_0^3 = 3(3^2) - \frac{2}{3}(3^3) = 27 - 18 = 9.
\]

**Alternate solution (outline):** The problem can be done using vertical slices, but this approach is much more complicated. One must divide the region into three parts, and find the values...
of $x$ at the rightmost and leftmost points in the region. One gets
\[
\int_{-4}^{-3} [(2 + \sqrt{4+x}) - (2 - \sqrt{4+x})] \, dx + \int_{-3}^{0} [(1 + \sqrt{1-x}) - (2 - \sqrt{4+x})] \, dx \\
+ \int_{0}^{1} [(1 + \sqrt{1-x}) - (1 - \sqrt{1-x})] \, dx \\
= \int_{-4}^{-3} 2\sqrt{4+x} \, dx + \int_{-3}^{0} (\sqrt{4+x} + \sqrt{1-x} - 1) \, dx + \int_{0}^{1} 2\sqrt{1-x} \, dx.
\]
The rest of the calculation is omitted.

8. (15 points.) Professor Greenbottle was doing an experiment on a sample of plutonium. The temperature of his sample, measured in $^\circ$K, at time $t$ hours after the beginning of his experiment, is modelled by
\[
T(t) = \frac{4}{\sqrt{2t+1}} + 3te^{-t},
\]
for $0 \leq t \leq 32$. (At time $t = 32$, the sample exploded, putting an end to the experiment.) According to the model, what was average temperature of the sample between 3 and 5 hours after the start of the experiment?

**Solution:** We need to find the average value of $T(t)$ for $t$ in $[3, 5]$, which is
\[
\frac{1}{5-3} \int_{3}^{5} T(t) \, dt = \frac{1}{2} \left( \frac{4}{\sqrt{2t+1}} dt + \int_{3}^{5} 3te^{-t} \, dt \right).
\]
We do the two integrals separately. For the first, we use the substitution $u = 2t+1$, so $du = 2 \, dt$ and $4 \, dt = 2 \, du$, to get
\[
\int \frac{4}{\sqrt{2t+1}} \, dt = \int 4(2t+1)^{-1/2} \, dt = \int 2u^{-1/2} \, du \\
= 2 \left( \frac{1}{(1/2)} \right) u^{1/2} + C = 4u^{1/2} + C = 4(2t+1)^{1/2} + C,
\]
so
\[
\int_{3}^{5} \frac{4}{\sqrt{2t+1}} \, dt = 4(2t+1)^{1/2} \bigg|_{3}^{5} = 4\sqrt{11} - 4\sqrt{7}.
\]
For the second, we integrate by parts, taking $u(t) = 3t$, $v'(t) = e^{-t}$, $u'(t) = 3$, and $v(t) = -e^{-t}$. (Getting $v(t)$ from $v'(t)$ requires the substitution $w = -t$.) This gives, at the second step again using the substitution $w = -t$,
\[
\int_{3}^{5} 3te^{-t} \, dt = (3te^{-t}) \bigg|_{3}^{5} - \int_{3}^{5} 3(-e^{-t}) \, dt = (-3te^{-t}) \bigg|_{3}^{5} - (3e^{-t}) \bigg|_{3}^{5} \\
= -15e^{-5} - (-9e^{-3}) - 3e^{-5} - (-3e^{-3}) = 12e^{-3} - 18e^{-5}.
\]
So the average temperature was 
\[
\frac{1}{5-3} \left( \int_3^5 \frac{4}{\sqrt{2t+1}} \, dt + \int_3^5 3te^{-t} \, dt \right) = \frac{1}{2} \left( 4\sqrt{11} - 4\sqrt{7} + 12e^{-3} - 18e^{-5} \right) = 2\sqrt{11} - 2\sqrt{7} + 6e^{-3} - 9e^{-5},
\]
in °K.
(The units in the final answer are \textit{required}.)

9. (15 points.) Define 
\[
p(z) = \int_{z^3-z}^{-2} t\sqrt{63 + \cos(t)} \, dt.
\]
Find \(p'(z)\).

\textit{Solution:} Define 
\[
F(x) = \int_{-2}^x t\sqrt{63 + \cos(t)} \, dt.
\]
Then (using the Fundamental Theorem of Calculus for the second formula) 
\[
p(z) = F(z^3-z) \quad \text{and} \quad F'(x) = x\sqrt{63 + \cos(x)}.
\]
Using the chain rule, we get 
\[
g'(z) = \frac{d}{dz} \left( F(z^3-z) \right) = F'(z^3-z) \frac{d}{dz} (z^3-z)
\]
\[
= (z^3-z)\sqrt{63 + \cos(z^3-z)} \cdot (3z^2-1) = (3z^2-1)(z^3-z)\sqrt{63 + \cos(z^3-z)}.
\]
There is no need to multiply out the expression \((3z^2-1)(z^3-z)\), but, if you do, you get 
\[
(3z^5 - 4z^3 + z)\sqrt{63 + \cos(z^3-z)}.
\]
(The variable \(t\) may not appear anywhere in the answer, because \(p'(z)\) must be a function of \(z\).)

10. (15 points.) Solve the initial value problem 
\[
y'(t) = 6t[y(t)]^2, \quad y(2) = -\frac{1}{15}.
\]

\textit{Solution:} We find the general solution. The function \(y(t) = 0\) for all \(t\) is a solution, but not the one we want. Otherwise, since solution curves don’t cross, \(y(t) \neq 0\) for all \(t\).
So we can rewrite the equation as 
\[
y'(t)y(t)^{-2} = 6t.
\]
Now integrate:
\[
\int y(t)^{-2}y'(t) \, dt = \int 6t \, dt.
\]
We have $\int 6t \, dt = 3t^2 + C_2$. On the left, use the substitution $u = y(t)$, so $du = y'(t) \, dt$, to get

$$\int y(t)^{-2}y'(t) \, dt = u^{-2} \, du = -u^{-1} + C_1 = -y(t)^{-1} + C_1.$$ 

Combine these, and take $C = C_2 - C_1$, to get

$$-y(t)^{-1} = 3t^2 + C.$$ 

Solve for $y(t)$:

$$y(t) = -\frac{1}{3t^2 + C},$$

for an arbitrary constant $C$.

We now want $y(1) = -\frac{1}{15}$. We write

$$-\frac{1}{15} = y(2) = -\frac{1}{3 \cdot 2^2 + C} = -\frac{1}{12 + C}.$$ 

So $C = 3$, and the solution is

$$y(t) = -\frac{1}{3t^2 + 3}.$$ 

11. (15 points.) Consider the region between the curve $y = \sqrt{2x}$ and the lines $x = 8$ and $y = 2$. It is rotated about the $x$-axis. Find the volume of the resulting solid.

**Solution:** Here are three pictures. On the right is a graph showing the lines and curve bounding the region to be rotated, and (dotted lines) the reflections of the curve and horizontal line in the $x$-axis (the axis of rotation). The other two pictures are two views of the resulting solid of revolution (opaque), the first seen from smaller values of $x$ and the second seen from larger values of $x$.

Here are three pictures related to methods for finding the volume, all with semitransparent versions of the solid. The left and middle pictures are two views of a washer shaped slice (at $x = 6$). The right picture shows a cylindrical shell (at $y = 3$).
We use vertical washers, which give an integral with respect to $x$. One limit of integration is $x = 8$. The other limit of integration is $x$-coordinate of the point at which the line $y = 2$ intersects the curve $y = \sqrt{2x}$. So we solve $2 = \sqrt{2x}$, which has the unique solution $x = 2$.

For $2 \leq x \leq 8$, a cross section at $x$ and perpendicular to the $x$-axis is an annulus with inner radius 2 and outer radius $\sqrt{2x}$. So its area is

$$\pi(\sqrt{2x})^2 - \pi(2)^2 = 2\pi x - 4\pi.$$ 

If it has thickness $\Delta x$ for small $\Delta x$, its volume is approximately $(2\pi x - 4\pi) \Delta x$. This leads to the integral

$$\int_2^8 (\pi(\sqrt{2x})^2 - \pi(2)^2) \, dx = \int_2^8 (2\pi x - 4\pi) \, dx$$

$$= (\pi x^2 - 4\pi x) \bigg|_2^8 = (64\pi - 32\pi) - (4\pi - 8\pi) = 36\pi.$$ 

The justification for using the washer method to set up the integral is not required for full credit for the solution, since it repeats what is in the textbook. It is included to reinforce the general approach to recognizing how to set up integrals in applications.

Alternate solution (sketch): We use cylindrical shells with horizontal axis, which gives an integral with respect to $y$. One limit of integration is $y = 2$. The other is the $y$-coordinate of the point at which the line $x = 8$ intersects the curve $y = \sqrt{2x}$. So we find $y = \sqrt{2 \cdot 8} = 4$. We also need to rewrite the equation $y = \sqrt{2x}$ as $x = \frac{1}{2}y^2$.

Consider a height $y$ with $2 \leq y \leq 4$. The cylindrical shell at this value of $y$ has radius $y$ and length $8 - \frac{1}{2}y^2$ (the difference of values of $x$). So its area is $2\pi y(8 - \frac{1}{2}y^2)$. If it has thickness $\Delta y$, then it has volume approximately $2\pi y(8 - \frac{1}{2}y^2) \Delta y$. This leads to the integral

$$\int_2^4 2\pi y \left(8 - \frac{1}{2}y^2\right) \, dy = \int_2^4 \pi y(16 - y^2) \, dy.$$
As with the first solution, the justification for using the shell method to set up the integral is not required for full credit for the solution.

12. (15 points) Let $f$ be a function defined on $(-\infty, \infty)$, and assume that $f$, $f'$, and $f''$ are all continuous. Suppose that $f(4) = 300$, $f'(4) = -50$, and $f''(4) = e$, and that $f(9) = -2000$, $f'(9) = \sqrt{2}$, and $f''(9) = \pi$. Find the exact value of $\int_4^9 xf''(x) \, dx$. You might not need all the information provided.

Solution: We integrate by parts, taking $u = x$, $dv = f''(x) \, dx$, $du = dx$, and $v = f'(x)$. This gives (using the Fundamental Theorem of Calculus at the second step):

$$\int_4^9 xf''(x) \, dx = xf'(x) \bigg|_4^9 - \int_4^9 f'(x) \, dx = xf'(x) \bigg|_4^9 - f(x) \bigg|_4^9$$

$$= 9f'(9) - 4f'(4) - [f(9) - f(4)] = 9\sqrt{2} - 4(-50) + 2000 + 300 = 9\sqrt{2} + 2500.$$ 

13. (15 points.) Determine whether or not the improper integral $\int_1^\infty (8 - 3 \cos(10y^3))e^{-y} \, dy$ converges.

Solution: For all $t \neq 0$ we have $-3 \leq -3 \cos(10y^3) \leq 3$, so $5 \leq 8 - 3 \cos(10y^3) \leq 11$. So $0 \leq 5e^{-y} \leq (8 - 3 \cos(10y^3))e^{-y} \leq 11e^{-y}$. In particular, $0 \leq (8 - 3 \cos(10y^3))e^{-y} \leq 11e^{-y}$. Using the substitution $u = -y$, so $du = -dy$ and $dy = -du$, we get

$$\int e^{-y} \, dy = \int (-e^u) \, du = -e^u + C = -e^{-y} + C.$$ 

So

$$\int_1^\infty e^{-y} \, dy = (-e^{-y}) \bigg|_1^\infty = \lim_{y \to \infty} (-e^{-y} + e^{-1}) = e^{-1} = \frac{1}{e}.$$ 

Thus $\int_1^\infty e^{-y} \, dy$ converges. Therefore the Comparison Test shows that $\int_1^\infty (8 - 3 \cos(10y^3))e^{-y} \, dy$ converges.

14. (15 points.) A reservoir on the planet Qzork is to be impounded by a dam in a narrow valley. The dam will be vertical, triangular, flat and 60 meters wide at the top, and narrowing to a point at the bottom of the valley 20 meters below. The acceleration due to gravity on the surface of Qzork is exactly $5 \, \text{m/sec}^2$. Find the total force exerted on the dam by the reservoir when it is full of water (density $1000 \, \text{kg/m}^3$).
Solution: Slice the dam into thin horizontal strips. Consider the strip at height $h$ meters above the bottom of the valley, and with the width (height) of the strip being $\Delta h$. The length is $3h$ meters. (Length is linear in $h$, is 0 when $h = 0$, and is 60 when $h = 20$.) So the area is approximately $3h\Delta h$. The force on such an area is $g$ times the mass of the water above a horizontal region of the same area. The volume of that amount of water is $(20 - h) \cdot 3h\Delta h$. (The first factor is the depth of the water.) So the force on our narrow strip is $1000g(20 - h) \cdot 3h\Delta h$. Therefore the total force is

$$\int_0^{20} 3000g(20 - h)dh = 3000g \int_0^{20} (20 - h^2)dh = 3000g \left(10h^2 - \frac{1}{3}h^3\right)\bigg|_0^{20} = (4 \cdot 10^6)g.$$ 

Putting in the value $g = 5$ and the units, we get $2 \cdot 10^7$ m kg/sec$^2$. The units can be rewritten as Newton-meters (N-m), or as joules (J).

15. (13 points.) The city of Megalopolis is circular and has a radius of 9 miles. The market value of land located $r$ miles from the city center is $e^{-r/3}$ billion dollars per square mile. Set up an integral which represents the total market value of all the land in Megalopolis. Include an explanation. Don’t try to evaluate the integral.

Solution: Consider a thin annulus (ring shaped area) at distance $r$ from the center and with width $\Delta r$. It has area approximately $2\pi r \Delta r$, and therefore has a total market value of approximately $e^{-r/3} \cdot 2\pi r \Delta r$ billion dollars. The corresponding integral is

$$\int_0^9 e^{-r/3} \cdot 2\pi r dr.$$ 

Extra credit. (25 extra credit points.) (Do not attempt this problem until you have finished all the ordinary problems on the exam and checked your answers to them. This problem will only be counted if you get at least 150 points on the main part of this exam.)

Find $\int \frac{4}{u(u-1)(u-2)} du$.

Solution: We do this integral using partial fractions. We need to find $a$, $b$, and $c$ such that

$$(1) \quad \frac{4}{u(u-1)(u-2)} = \frac{a}{u} + \frac{b}{u-1} + \frac{c}{u-2}$$

for all real numbers $u \neq 0, 1, 2$. Now

$$\frac{a}{u} + \frac{b}{u-1} + \frac{c}{u-2} = \frac{a(u-1)(u-2) + bu(u-2) + cu(u-1)}{u(u-1)(u-2)} = \frac{(a + b + c)u^2 + (3a - 2b - c)u + 2a}{u(u-1)(u-2)}.$$

If (1) is to be true for all real $u$ for which the denominator is not zero, we must have

$$a + b + c = 0, \quad -3a - 2b - c = 0, \quad \text{and} \quad 2a = 4.$$
So $a = 2$. Substituting this in the other two equations gives 

$$b + c = -2$$

and

$$2b + c = -6.$$  

These two equations have the solution $b = -4$ and $c = 2$. Therefore

$$\frac{4}{u(u - 1)(u - 2)} = \frac{2}{u} - \frac{4}{u - 1} + \frac{2}{u - 2}.$$  

We integrate, using the substitution $w_2 = u - 1$ (so $dw_2 = du$) on the second term and the substitution $w_3 = u - 2$ (so $dw_3 = du$) on the third term, getting

$$\int \frac{4}{u(u - 1)(u - 2)} \, du = \int \frac{2}{u} \, du - \int \frac{4}{u - 1} \, du + \int \frac{2}{u - 2} \, du$$

$$= \int \frac{2}{u} \, du - \int \frac{4}{w_2} \, dw_2 + \int \frac{2}{w_3} \, dw_3$$

$$= 2 \ln(|u|) - 4 \ln(|w_2|) + 2 \ln(|w_3|) + C$$

$$= 2 \ln(|u|) - 4 \ln(|u - 1|) + 2 \ln(|u - 2|) + C.$$

**Alternate solution:** This solution differs only in the method used to find the coefficients $a$, $b$, and $c$ in the partial fraction expansion. We need $a$, $b$, and $c$ satisfying (1) for all real numbers $u \neq 0, 1, 2$. As in the first solution, this is the same as

$$\frac{4}{u(u - 1)(u - 2)} = \frac{a(u - 1)(u - 2) + bu(u - 2) + cu(u - 1)}{u(u - 1)(u - 2)}$$  

for all real numbers $u \neq 0, 1, 2$. This happens if and only if

(2)  

$$4 = a(u - 1)(u - 2) + bu(u - 2) + cu(u - 1)$$  

for all real numbers $u \neq 0, 1, 2$. Now both sides of (2) are continuous functions of $u$. Therefore, if (2) holds for all real numbers $u \neq 0, 1, 2$, then (2) also holds for $u = 0, 1, 2$. Putting $u = 0$ in (2) gives

$$4 = a(-1)(-2),$$

putting $u = 1$ in (2) gives

$$4 = b(1)(-1),$$

and putting $u = 2$ in (2) gives

$$4 = c(2)(1).$$

So $a = 2$, $b = -4$, and $c = 2$. Now finish as in the first solution.