#4. Average value of \( f(x) = \frac{8e^x}{\sqrt{1+ae^x}} \) on \([b, b+\ln(a)],[b, b+\ln(a)]\) with \( a \in \mathbb{R}^+ \).

\[ \text{Solution:} \quad f \equiv \frac{1}{b(b+\ln(a))} \int_{b(b+\ln(a))}^{b(b+\ln(a))} \frac{8e^x}{\sqrt{1+ae^x}} \, dx \]

\[ \text{Do} \, f(x) \Rightarrow \text{Substitute} \quad u = 1+ae^x, \quad \text{so} \quad \frac{du}{dx} = ae^x \text{,} \quad \text{so} \quad dx = \frac{1}{ae^x} \, du. \]

\[ \int \frac{8e^x}{\sqrt{1+ae^x}} \, dx = \int \left( \frac{8e^x}{\sqrt{u}} \right) \left( \frac{1}{ae^x} \right) \, du = \frac{8}{a} \int \frac{\sqrt{u}}{u} \, du = \frac{8}{a} \int u^{-1/2} \, du \]

\[ = \frac{8}{a} \left( \frac{1}{\frac{1}{2}} \right) u^{1/2} + C = \frac{16}{a} (1+ae^x)^{1/2} + C \]

\[ \int_b^{b+\ln(a)} \frac{8e^x}{\sqrt{1+ae^x}} \, dx = \frac{16}{a} (1+ae^x)^{1/2} \bigg|_b^{b+\ln(a)}} \]

\[ \text{Note: Simplify} \ e^{\ln(x)} = x \text{, if we define} \ C = -C_0 \]

#2. \( \int G(x) \, dx \quad \text{given} \quad G'(x) = 9x \sec^2(x^3) \)

\[ \text{We let:} \quad u = G(x), \quad v = x \]

\[ du = G'(x) \, dx, \quad dv = x \, dx \]

\[ = u(x) \frac{x}{x^2} - \int v \, du = G(x) - \int 9x \sec^2(x^3) \, dx = xG(x) - \int 9x^3 \sec^2(x^3) \, dx. \]

Now substitute \( w = x^3 \), so \( dw = 3x^2 \, dx \), and \( 9x^2 \, dx = 3dw \).

\[ \int 9x^2 \sec^2(x^3) \, dx = \int 3 \sec^2(w) \, dw = 3 \tan(w) + C_0, \] \( \text{Revert:} \)

\[ \int G(x) \, dx = xG(x) - \int 9x^3 \sec^2(x^3) \, dx = xG(x) - \left[ 3 \tan(x^3) + C_0 \right] = xG(x) - 3 \tan(x^3) + C \]

\[ \text{For unknown} \ a \text{ and } b, \text{ which you need to find.} \]

#6. \( \int \frac{12}{(x-2)(x+1)} \, dx \)

\[ \text{Method? Partial fraction.} \]

Write \( \frac{12}{(x-2)(x+1)} = \frac{a}{x-2} + \frac{b}{x+1} \)

To find \( a \) and \( b \), combine fractions on right:

\[ \frac{12}{(x-2)(x+1)} = \frac{a(x+1)}{(x-2)(x+1)} + \frac{b(x-2)}{(x-2)(x+1)} = \frac{a(x+1) + b(x-2)}{(x-2)(x+1)} \]

\[ a(x+1) + b(x-2) \]

\[ a(x+1) + b(x-2) = 12 \]

\[ a + a + 2b \]

\[ a + b + 2 \]
\[ \text{HINT: for all } x, \quad 12 = (a+b)x + a - 2b. \]

So, \( b = -a \), and \( a - 2(1-a) = 12 \), so \( 3a = 12 \).

\[ a = 4, \quad b = -4, \]

and

\[ \frac{12}{(x+2)(x+1)} = \frac{4}{x+2} - \frac{4}{x+1} \]

So, \( \int \frac{12}{(x+2)(x+1)} \, dx = \int \frac{4}{x+2} \, dx - \int \frac{4}{x+1} \, dx \)

\[ \text{Both done by sub, } u_1 = x+2, \quad u_2 = x+1 \quad \text{etc.} \]

\[ \#12 \int (1 + 4t^2)^{7/2} \, dt \]

If \( u = 1 + 4t^2 \), then \( du = 8t \, dt \), and \( \int \frac{1}{4} u^{3/2} \, du \) (can be fixed).

Use trig sub. For square roots of quadratic polynomials:

Also for \( \frac{1}{1 + x^2} \), \( \frac{1}{(1 + x^2)^2} \), etc.,

Half integer powers

Whose aren't worked by partial fractions since \( x \) and \( x^2 \) in denominator.

Three possible substitutions to consider:

- \( t = a \sin \theta, \quad t = a \tan \theta, \quad t = a \sec \theta \)

Suppose try \( t = a \sin \theta \). Then \( 1 + 4t^2 = 1 + 4a^2 \sin^2 \theta \). Can't be simplified

(try a trig identity) \( \text{??} \)

Suppose try \( t = a \tan \theta \). Then \( 1 + 4t^2 = 1 + 4a^2 \tan^2 \theta \), Can't be simplified?

Should have \( a = \frac{1}{2} \), so \( \int 1 + 4 \left( \frac{1}{2} \right)^2 \tan^2 \theta = 1 + \tan^2 \theta = \sec^2 \theta \).

With \( \theta = \frac{1}{2} \sin \theta \), so \( 1 + \sin^2 \theta \). No trig identity helps here.

\[ \int (1 + 4t^2)^{7/2} \, dt \]

\[ t = \frac{1}{2} \tan \theta, \quad \text{so} \quad dt = \frac{1}{2} \sec^2 \theta \, d\theta, \]

\[ \int \sec^2 \theta \cdot \frac{1}{2} \sec^2 \theta \, d\theta = \frac{1}{2} \sec^2 \theta \, d\theta \]
One molecule of A and one molecule of B collide to make one molecule of C.

Reaction rate proportional to product of concentrations, initial concentrations are \(a_0\), \(b_0\). Write diffusion for concentration of C, given that there is none present at the beginning.

Give names to things. Let \(a(t)\) be the concentration of compound A

\[ a(t) \]

\[ b(t) \]

\[ c(t) \]

Then \( c'(t) = k \ a(t) \ b(t) \) for some positive constant \(k\). (Want to replace \(a(t)\) and \(b(t)\) with something involving \(c(t)\).

\[
\left[ \text{# of molecules of A at time } t \right] \times \left[ \text{# of molecules of C at time } t \right] = \text{# of molecules of A at time } t.
\]

In terms of concentrations, set \( a(t) + c(t) = a_0 \). So \( a(t) = a_0 - c(t) \).

Similarly, doing the same for B gives: \( b(t) + c(t) = b_0 \), so \( b(t) = b_0 - c(t) \). Now eqn.

becomes \( c'(t) = k \ (a_0 - c(t)) \ (b_0 - c(t)) \), for an unknown positive constant \(k\).

To solve (not asked for in problem): This eqn. is separable:

\[
\frac{c'(t)}{(a_0 - c(t))(b_0 - c(t))} = \frac{1}{k} \int dt
\]

Doing this uses partial fractions.