SOLUTIONS TO THE FINAL EXAM SAMPLE PROBLEMS

The final exam is comprehensive. It will probably have greater emphasis on material covered since Midterm 2. Formally there may appear to be fewer problems on earlier material, since earlier material is often used in problems on later material.

There is only one attempt at the final exam, unlike for the midterms.

Most problems will be similar to WeBWorK problems, written homework problems, and the problems here. Note, though, that the exact form of functions which appear could vary substantially.

Be sure to get the notation right! (This is a frequent source of errors.) The right notation will help you get the mathematics right, and I will complain about incorrect notation.

The problems here have point values attached, which give a very rough idea of the point values problems requiring a similar amount of work will have on the real exam. Note, however, that the real midterm will total 200 points, and the problems here total somewhat more than that.

The final exam from the last time I taught the course should give a reasonable idea of the length to expect.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation in the book, in handouts, in files posted on the course website, and on the blackboard; use it. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Instructions for the final exam via Zoom are on the course home page, linked at:

https://pages.uoregon.edu/ncp/Courses/Math252_W21_Web/FinalExam/FinalExam.html.
(This link should be clickable.) The main difference from those for Midterm 2 is the length and time. Remember that work must be written in an organized and logical manner, and must say in correct notation what you did. This includes correct use of parentheses, writing equals signs when you are saying two things are equal, and much more; see:

https://pages.uoregon.edu/ncp/Courses/Math252_W21_Web/CourseInfo/GenHWInstr.pdf
and

https://pages.uoregon.edu/ncp/Courses/Math252_W21_Web/CourseInfo/Notation/Notation.html.
(These links should be clickable.)

Here are the mathematical instructions for the final exam:

(1) The point values are as indicated in each problem; total 200 points (not counting extra credit).

Date: 19 March 2021.
2. Show enough of your work that your method is obvious. Be sure that every statement you write is correct. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.

3. Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.

4. When exact values are specified, give answers such as $\frac{1}{7}$, $\sqrt{2}$, $\ln(2)$, or $\frac{2\pi}{9}$. Calculator approximations will not be accepted.

5. Final answers must always be simplified unless otherwise specified.

Extra credit will be given to the first two people catching any error in the sample problems or their solutions, provided I am notified before corrections are posted. You must specify both the error and the correct version.

1. **Problems on new material**

This section consists of sample problems on material covered since Midterm 2.

1. (8 points.) Determine whether or not the function $y(t) = 6e^{2t} - e^t$ satisfies the differential equation $y'(t) = 2y(t) + e^t$. Show your work.

   **Solution:** We check:
   
   $$y'(t) = 12e^{2t} - e^t$$
   
   and
   
   $$2y(t) + e^t = 2(6e^{2t} - e^t) + e^t = 12e^{2t} - 2e^t + e^t = e^{2t} - e^t.$$
   
   Therefore the given function solves the equation.

2. (8 points.) Determine whether or not the function $y(t) = 3e^{2t} + e^t$ satisfies the differential equation $y'(t) = 2y(t) + e^t$. Show your work.

   **Solution:** We check:
   
   $$y'(t) = 6e^{2t} + e^t$$
   
   and
   
   $$2y(t) + e^t = 2(3e^{2t} + e^t) + e^t = 6e^{2t} + 3e^t.$$
   
   For $t = 0$ we get
   
   $$y'(0) = 6e^0 + e^0 = 7$$
   
   and
   
   $$2y(0) + e^0 = 6e^0 + 3e^0 = 9.$$
   
   So $y'(0) \neq 2y(0) + e^0$. Therefore the given function does not solve the equation.

   For full credit, a solution must exhibit a specific value of $t$ (such as $t = 0$ above; many other choices work) such that $y'(t) \neq 2y(t) + 4t^2$. 

3. (15 points.) A chemical reaction in a solution involves combining three molecules of compound A to produce a molecule of compound X. The reaction occurs when three molecules of compound A all collide at the same time, so the concentration $a(t)$ of compound A at time $t$ is governed by the differential equation

$$a'(t) = -ka(t)^3$$

for some constant $k > 0$.

Suppose that $t$ is measured in hours, that $a(t)$ is measured in moles per liter, that when the reaction begins the concentration of compound A is 1 mole per liter, and that 2 hours later the concentration of compound A is $\frac{1}{2}$ moles per liter. Find $k$.

Solution: We solve the differential equation. The function $a(t) = 0$ for all $t$ is a solution, but it is obviously not the one we want. For other solutions, $a(t)$ is never zero, so we can rewrite the equation as

$$\frac{a'(t)}{a(t)^3} = -k.$$

Now integrate:

$$\int \frac{a'(t)}{a(t)^3} \, dt = \int (-k) \, dt.$$

We have $\int (-k) \, dt = -kt + C_2$. On the left, use the substitution $u = a(t)$, so $du = a'(t) \, dt$, to get

$$\int \frac{a'(t)}{a(t)^3} \, dt = \int u^{-3} \, du = -\frac{1}{2}u^{-2} + C_1 = -\frac{1}{2a(t)^2} + C_1.$$

Combine these, and take $C = C_2 - C_1$, to get

$$-\frac{1}{2a(t)^2} = -kt + C.$$

Solve for $a(t)$:

$$a(t) = \frac{2}{\sqrt{kt - C}},$$

for an arbitrary constant $C$.

We take $t = 0$ to be the starting time of the reaction. Then $a(0) = 1$ and $a(2) = \frac{1}{2}$. Thus

$$1 = a(0) = \frac{2}{\sqrt{k \cdot 0 - C}} = \frac{2}{\sqrt{-C}},$$

and

$$\frac{1}{2} = a(2) = \frac{2}{\sqrt{k \cdot 2 - C}} = \frac{2}{\sqrt{2k - C}}.$$

The first equation implies that $\sqrt{-C} = 2$, so $C = -4$. The second equation now implies that

$$\frac{1}{2} = \frac{2}{\sqrt{2k - 4}}.$$

Therefore $2k + 4 = 4^2 = 16$, so $k = 6$.

The correct units for $k$ are determined by the fact that $a'(t)$ is measured in (moles/liter)/minute and $a(t)^3$ is measured in (moles/liter)$^3$. So $k$ is $6$ liters$^2$ per mole$^2$-minute.
4. (18 points.) Find all solutions to the differential equation

\[ y'(x) = \frac{2y(x)^2}{1 + x^2}. \]

Find the solution \( y(x) \) satisfying \( y(0) = \frac{1}{6} \), and for this solution determine \( \lim_{x \to \infty} y(x) \).

**Solution:** First, \( y(x) = 0 \) for all \( x \) is a solution. For other solutions, \( y(x) \) is never 0, so we can rewrite the equation as

\[ \frac{y'(x)}{y(x)^2} = \frac{2}{1 + x^2}. \]

Now integrate:

\[
\int \frac{y'(x)}{y(x)^2} \, dx = \int \frac{2}{1 + x^2} \, dx.
\]

We have \( \int \frac{2}{1 + x^2} \, dx = 2 \arctan(x) + C_2 \). On the left, use the substitution \( u = y(x) \), so \( du = y'(x) \, dx \), to get

\[
\int \frac{y'(x)}{y(x)^2} \, dx = \int u^{-2} \, du = -u^{-1} + C_1 = -\frac{1}{y(x)} + C_1.
\]

Combine these, and take \( C = C_2 - C_1 \), to get

\[ -\frac{1}{y(x)} = 2 \arctan(x) + C. \]

Solve for \( y(x) \):

\[ y(x) = -\frac{1}{2 \arctan(x) + C}, \]

for an arbitrary constant \( C \).

We want \( y(0) = \frac{1}{6} \). Write

\[ -\frac{1}{2 \arctan(0) + C} = \frac{1}{6}; \]

and solve for \( C \). Since \( \arctan(0) = 0 \), the solution is \( C = -6 \). So

\[ y(x) = -\frac{1}{2 \arctan(x) - 6} = \frac{1}{6 - 2 \arctan(x)}. \]

We have

\[ \lim_{x \to \infty} y(x) = \lim_{x \to \infty} \frac{1}{6 - 2 \arctan(x)} = \frac{1}{6 - \lim_{x \to \infty} 2 \arctan(x)} = \frac{1}{6 - \pi}. \]

5. (12 points.) Find all solutions to the differential equation \( y'(x) = \cos(x)[2y(x) - 4]^2 \).

**Solution:** First, \( y(x) = 2 \) for all \( x \) is a solution. For other solutions, \( y(x) \) is never 2, so we can rewrite the equation as

\[ \frac{y'(x)}{[2y(x) - 4]^2} = \cos(x). \]

Now integrate:

\[
\int \frac{y'(x)}{[2y(x) - 4]^2} \, dx = \int \cos(x) \, dx.
\]
We have \( \int \cos(x) \, dx = \sin(x) + C_2 \). On the left, use the substitution \( u = 2y(x) - 4 \), so \( du = 2y'(x) \, dx \), to get
\[
\int \frac{y'(x)}{(2y(x) - 4)^2} \, dx = \int \frac{1}{2} u^{-2} \, du = -\frac{1}{2} u^{-1} + C_1 = -\frac{1}{2(2y(x) - 4)} + C_1.
\]
Combine these, and take \( C = C_2 - C_1 \), to get
\[
-\frac{1}{2(2y(x) - 4)} = \sin(x) + C.
\]
Solve for \( y(x) \):
\[
y(x) = \frac{1}{4(- \sin(x) - C)} + 2 = 2 - \frac{1}{4(\sin(x) + C)},
\]
for an arbitrary constant \( C \).

Alternate solution (only the notation is different): The book writes this in “differential notation”, which is a formal version of the above:
\[
\frac{dy}{dx} = \cos(x)(2y - 4)^2
\]
\[
\int \frac{1}{(2y - 4)^2} \, dy = \int \cos(x) \, dx
\]
\[
\int \frac{1}{(2y - 4)^2} \, dy = \int \cos(x) \, dx
\]
\[
-\frac{1}{2(2y - 4)} + C_1 = \sin(x) + C_2
\]
\[
-\frac{1}{2(2y - 4)} = -\sin(x) + C
\]
\[
2(2y - 4) = -\frac{1}{\sin(x) + C}
\]
\[
y = 2 - \frac{1}{4(\sin(x) + C)},
\]
for an arbitrary constant \( C \).

6. (8 points.) In the absence of other effects, the population of lemmings on an island grows at a rate proportional to the existing population. In addition, there is a wolf on this island, which eats 100 lemmings per year. (The wolf can’t reproduce, since there is only one.) Moreover, 20 lemmings per year swim to a neighboring island to escape from the wolf.

Set up a differential equation which models the population of lemmings on this island as a function of time. Be sure to state what your variables mean. You will have one unknown constant in your equation; state that it is unknown constant, and state whether it is positive or negative.
Solution: Let $L(t)$ be the population of lemmings on the island, with $t$ measured in years. There are three contributions to the rate of change of $M(t)$. The first is a positive contribution from the growth rate, which is $kL(t)$ for some unspecified constant $k > 0$. The second is a negative contribution by the wolf, which is $-100$. The third is a negative contribution from lemmings swimming to the neighboring island, which is $-20$. So the equation is

$$L'(t) = kL(t) - 100 - 20 = kL(t) - 120,$$

for some unspecified constant $k > 0$.

7. (10 points.) Here is a direction field:

On this graph, draw the graphs of the five solutions $y(x)$ to the equations satisfying the initial conditions $y(0) = -2, -1, 0, 1, 2$.

Solution: Here are the requested solution curves.
In case anyone is interested, the differential equation used to produce this direction field is 
\[ y'(x) = \frac{(xy(x) + 1)}{(x^2 + 2)}. \]

8. (10 points.) A vat in a campus bar starts out at the beginning of finals week with 200 liters of orange juice. Vodka is added to it at 40 liters per day, the contents are kept well mixed, and 40 liters per day of the mixture are sold to students who have finished their final exams. (So the students finishing later get more vodka.)

Set up, but do not solve, a differential equation which models the amount of vodka in the vat as a function of time. Be sure to state what your variables mean.

**Solution:** Let \( t \) be time, measured in days. Let \( V(t) \) be the total amount of vodka in the vat at time \( t \), measured in liters. There are two contributions to the rate of change of the total amount of vodka in the vat: inflow and outflow.

Inflow gives the contribution 40 liters per day. For outflow, each liter of the mixture contains \( \frac{V(t)}{200} \) liters of vodka. Since 40 liters of the mixture are removed per day, it gives the contribution \( -\frac{40V(t)}{200} \) liters of vodka per day. We get the differential equation

\[
V'(t) = 40 - \frac{40V(t)}{200} = 40 - \frac{V(t)}{5}.
\]

9. (15 points.) Consider the differential equation

\[ y'(x) = kxe^{-y(x)}, \]

in which \( k \) is a constant. Suppose a solution \( x \mapsto y(x) \) to this equation satisfies \( y(0) = 0 \) and \( y(2) = \ln(5) \). Find \( k \).
Solution: We find the general solution to the equation. It is separable. Since $e^{y(x)}$ is never zero, we can rewrite it as 

$$e^{y(x)}y'(x) = kx.$$ 

Now integrate:

$$\int e^{y(x)}y'(x) \, dx = \int k \, dx.$$ 

We have $\int k \, dx = \frac{1}{2}kx^2 + C$. On the left, use the substitution $u = y(x)$, so $du = y'(x) \, dx$, to get 

$$\int e^{y(x)}y'(x) \, dx = \int e^u \, du = e^u + C_1 = e^{y(x)} + C_1.$$ 

Combine these, and take $C = C_2 - C_1$, to get 

$$e^{y(x)} = \frac{1}{2}kx^2 + C.$$ 

Solve for $y(x)$: 

$$y(x) = \ln\left(\frac{1}{2}kx^2 + C\right),$$ 

for an arbitrary constant $C$.

We now have 

$$0 = y(0) = \ln\left(\frac{1}{2}k \cdot 0^2 + C\right) = \ln(C) \quad \text{and} \quad \ln(5) = y(2) = \ln\left(\frac{1}{2}k \cdot 2^2 + C\right) = \ln(2k+C).$$ 

The first equation implies that $C = 1$. The second equation now implies that $\ln(5) = \ln(2k+1)$, so $2k + 1 = 5$, whence $k = 2$.

The solution $y(x)$ in the problem is now given by $y(x) = \ln(x^2 + 1)$. (The problem didn’t ask for this.)

10. (10 points.) Consider the differential equation 

$$f'(x) = \frac{\sin(x)}{7 + \sin(f(x))}.$$ 

Give an equation which implicitly determines the general solution to this equation.

Solution: This equation is separable. Since 

$$\frac{1}{7 + \sin(f(x))}$$ 

is never zero, we can rewrite it as 

$$(7 + \sin(f(x)))f'(x) = \sin(x).$$ 

Now integrate: 

$$\int (7 + \sin(f(x)))f'(x) \, dx = \sin(x) \, dx.$$
On the right, we get $-\cos(x)$. On the left, use the substitution $u = f(x)$, so $du = f'(x)\,dx$, to get
\[ \int (7 + \sin(f(x))) f'(x)\,dx = \int (7 + \sin(u))\,dw = 7u - \cos(u) + C_1 = 7f(x) - \cos(f(x)) + C_1. \]
Combine these, and take $C = C_2 - C_1$, to get
\[ 7f(x) - \cos(f(x)) = -\cos(x) + C. \]
Stop here, since we are only asked for an equation which implicitly determines the general solution.

11. (12 points.) Solve the initial value problem
\[ y'(x) = \frac{x}{y(x)}, \quad y(1) = 2. \]

Solution: We find the general solution to the equation. It is separable. Since $1/y(x)$ is never zero, we can rewrite it as
\[ y(x)y'(x) = x. \]
Now integrate:
\[ \int y(x)y'(x)\,dx = \int x\,dx. \]
On the right, we get $\frac{1}{2}x^2 + C_2$. On the left, use the substitution $u = y(x)$, so $du = y'(x)\,dx$, to get
\[ \int y(x)y'(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}y(x)^2 + C_1. \]
Combine these, and take $C = C_2 - C_1$, to get
\[ \frac{1}{2}y(x)^2 = \frac{1}{2}x^2 + C. \]
Our initial condition says that, at least near $x = 1$, we are supposed to have $y(x) > 0$. When solving for $f(x)$, we therefore take the positive square root:
\[ y(x)^2 = x^2 + 2C, \]
so
\[ y(x) = \sqrt{x^2 + 2C}. \]
We now have
\[ 2 = y(1) = \sqrt{1 + 2C}. \]
So $4 = 1 + 2C$, which implies $2C = 3$, and the solution is $y(x) = \sqrt{x^2 + 3}$.

12. (10 points.) At its creation, a dungeons and dragons universe contains 100 goblins. The population grows at a rate proportional to the existing population. If the population of goblins doubles after 6 years, how long does it take until there are 500 goblins?
**Solution:** Let $P(t)$ be the population of goblins $t$ years after the creation of this universe. The problem tells us that there is a constant $k > 0$ such that $P'(t) = kP(t)$. We know the solutions to this differential equation: they are $P(t) = Ce^{kt}$ for arbitrary constants $C$.

We are given $P(0) = 100$, so $C = 100$. We are further told that $P(6) = 200$. So

$$200 = P(6) = 100e^{k\cdot6} = 100e^{6k},$$

whence $k = \frac{1}{6} \ln(2)$, and

$$P(t) = 100 \exp\left(\frac{\ln(2)t}{6}\right).$$

We need to find $t_0$ such that $P(t_0) = 500$. So we solve

$$500 = P(t_0) = 100 \exp\left(\frac{\ln(2)t_0}{6}\right),$$

which gives $\ln(5) = \ln(2)t_0/6$, so

$$t_0 = \frac{6 \ln(5)}{\ln(2)}.$$

So there are 500 goblins after $6 \ln(5)/\ln(2)$ years.

(This is the time at which this universe becomes too dangerous for the characters.)

2. **Problems on old material**

   This section contains a random selection of sample problems on material covered before Midterm 2. It is not necessarily expected to be a good match for the problems on old material on the real final exam. (It consists of the problems from the final exam the last time I taught this course which were on material from before Midterm 2.)

1. (13 points.) Find an antiderivative $F$ of the function $f(x) = x^2e^{x^3} - \sec^2(x) - 12x^2$ such that $F(0) = 2$.

   **Solution:** Using the substitution $u = x^3$, so $x^2 \, dx = \frac{1}{3} \, du$, we get

   $$\int x^2e^{x^3} \, dx = \int \frac{1}{3} e^u \, du = \frac{1}{3} e^u + C_1 = \frac{1}{3} e^{x^3} + C_1.$$

   We know directly that

   $$\int \sec^2(x) \, dx = \tan(x) + C_2 \quad \text{and} \quad \int 12x^2 \, dx = 4x^3 + C_3.$$

   So the general antiderivative of $f(x) = x^2e^{x^3} - \sec^2(x) - 12x^2$ is $F(x) = \frac{1}{3} e^{x^3} - \tan(x) - 4x^3 + C$. This gives $F(0) = \frac{1}{3} - 0 - 0 + C = \frac{1}{3} + C$. We want $F(0) = 2$, so we take $C = \frac{5}{3}$. Thus our answer is

   $$F(x) = \frac{1}{3} e^{x^3} - \tan(x) - 4x^3 + \frac{5}{3}.$$

2. (13 points.) Find $\int_0^3 12t \cos(2t) \, dt$. 
Solution: Integrate by parts with \( u(t) = t, \ v'(t) = 12 \cos(2t), \ u'(t) = 1, \) and \( v(t) = 6 \sin(2t). \) (To find \( v(t), \) we used the substitution \( w = 2t. \) ) Now, again using the substitution \( w = 2t \) at the second step, we get
\[
\int_0^3 12t \cos(2t) \, dt = t \cdot 6 \sin(2t) \bigg|_0^3 - \int_0^3 6 \sin(2t) \, dt = 6t \sin(2t) \bigg|_0^3 + 3 \cos(2t) \bigg|_0^3 = (6 \cdot 3) \sin(2 \cdot 3) - (6 \cdot 0) \sin(2 \cdot 0) + 3 \cos(2 \cdot 3) - 3 \cos(2 \cdot 0) = 18 \sin(6) + 3 \cos(6) - 3.
\]

Alternate solution (done by a student): Using the substitution \( w = 2t, \) so \( dt = \frac{1}{2} \, dw, \) we get
\[
\int_0^3 12t \cos(2t) \, dt = \int_0^6 3w \cos(w) \, dw.
\]
Now integrate by parts with \( u(w) = w, \ v'(w) = 6 \cos(w), \ u'(w) = 1, \) and \( v(w) = 6 \sin(w). \) This gives
\[
\int_0^6 3w \cos(w) \, dw = w \cdot 6 \sin(w) \bigg|_0^6 - \int_0^6 6 \sin(w) \, dw = 6w \sin(w) \bigg|_0^6 + 3 \cos(w) \bigg|_0^6 = (3 \cdot 6) \sin(6) - (3 \cdot 0) \sin(0) + 3 \cos(6) - 3 \cos(0) = 18 \sin(6) + 3 \cos(6) - 3.
\]

3. (10 points.) Consider the region between the curve \( y = x^3 + x + 1, \) the lines \( x = 1 \) and \( x = 4, \) and the \( x \)-axis. It is rotated about the \( y \)-axis. Set up an integral which gives the volume of the resulting solid. Do not evaluate the integral.

Solution: We use cylindrical shells with vertical axis. For \( 1 \leq x \leq 4, \) the cylindrical shell at this value of \( x \) and with thickness \( \Delta x \) has radius \( x \) and height \( x^3 + x + 1, \) hence volume approximately \( 2\pi x (x^3 + x + 1) \Delta x. \) This leads to the integral
\[
\int_1^4 2\pi x (x^3 + x + 1) \, dx.
\]

Using washers is not recommended. One must deal with the inverse of the function \( x \mapsto x^3 + x + 1. \)

4. (16 points.) Find \( \int \frac{6}{u(2-u)} \, du. \)

Solution: This integral is done by partial fractions.

We write
\[
\frac{6}{u(2-u)} = \frac{a}{u} + \frac{b}{2-u},
\]
and solve for \( a \) and \( b. \) For all \( t \neq 0, 2 \) we have
\[
\frac{6}{u(2-u)} = \frac{a}{u} + \frac{b}{2-u} = \frac{a(2-u)}{u(2-u)} + \frac{bu}{u(2-u)} = \frac{(-a+b)u+2a}{u(2-u)}.
\]
This can only happen if \(-a + b = 0\) and \(2a = 6\). These two equations have the simultaneous solution \(a = 3\) and \(b = 3\). Thus

\[
\frac{6}{u(2-u)} = \frac{3}{2-u} + \frac{3}{u},
\]

and

\[
\int \frac{6}{u(2-u)} \, du = \int \frac{3}{2-u} \, du + \int \frac{3}{u} \, du.
\]

Using the substitution \(v = 2 - u\), so \(du = -dv\), on the second integral, we have

\[
\int \frac{3}{u} \, du = 3 \ln(|u|) + C_1 \quad \text{and} \quad \int \frac{3}{2-u} \, du - 3 \ln(|2-u|) + C_2.
\]

So

\[
\int \frac{6}{u(2-u)} \, du = 3 \ln(|u|) + C_1 - 3 \ln(|2-u|) + C_2 = 3 \ln(|u|) - 3 \ln(|2-u|) + C.
\]

5. (12 points.) Define

\[h(x) = \int_6^{x^2} t \cos(t^5) \, dt.\]

Find \(h'(x)\).

**Solution:** Define

\[F(x) = \int_6^{x} t \cos(t^5) \, dt.\]

Then (using the Fundamental Theorem of Calculus for the second formula)

\[h(x) = F(x^2) \quad \text{and} \quad F'(x) = x \cos(x^5).\]

Using the chain rule, we get

\[h'(x) = \frac{d}{dx} (F(x^2)) = F'(x^2) \frac{d}{dx} (x^2) = x^2 \cos((x^2)^5) \cdot 2x = 2x^3 \cos(x^{10}).\]

(The variable \(t\) may not appear anywhere in the answer, because \(h'(x)\) must be a function of \(x\).)

6. (12 points.) A particle moves along a straight line in such a way that, \(t\) minutes after it starts moving, its distance to the right of its starting point is \(3 \sin(t) - 2t\) millimeters. During the period from 2 minutes to 5 minutes after its starting time, what is the average distance of the particle to the right of its starting point?
Solution: Let \( p(t) \) be the distance to the right of the starting point at time \( t \). We need to find the average value of \( p(t) \) for \( t \) in \([2, 5]\), which is

\[
\frac{1}{5-2} \int_2^5 p(t) \, dt = \frac{1}{3} \int_2^5 (3 \sin(t) - 2t) \, dt = \frac{1}{3} \left[ (-3 \cos(t) - t^2) \right]_2^5
\]

\[
= \frac{1}{3} (-3 \cos(5) - 5^2) - \frac{1}{3} (-3 \cos(2) - 2^2)
\]

\[
= -\frac{25 - 4}{3} + \cos(2) - \cos(5) = -7 + \cos(2) - \cos(5).
\]

So the average distance to the right of the starting point is \(-7 + \cos(2) - \cos(5)\) millimeters.

(The units in the final answer are \textit{required}.)

7. (12 points.) A small valley on the planet Yuggxth has a parabolic cross section: measured in meters, it looks like the region above the curve \( y = x^2 \). A vertical dam for a water reservoir is built in this valley. It is 9 meters high. When the reservoir is full, set up (but do not evaluate) an integral which gives the force it exerts on the dam. The density of water is 1000 kilograms per cubic meter, and the gravitational constant at the surface of Yuggxth is exactly 8 meters/second\(^2\).

Solution: Consider a horizontal strip at height \( h \) meters above the bottom of the valley, and such that the height (width) of the strip is \( \Delta h \) meters. The water at this height is \( 9 - h \) meters deep. The force on one square meter exerted by water of this depth is the volume of the water above a square meter, times the density of the water, times the acceleration due to gravity. These factors are \( 9 - h \) cubic meters, 1000 kilograms per cubic meter, and 8 meters/second\(^2\), giving a force of \((9 - h)(1000)(8)\) kilograms \cdot meters/second\(^2\). This is the force on one square meter, so the pressure is kilograms/[meters \cdot second\(^2\)].

The length of our horizontal strip is \( 2x \) when \( h = x^2 \), so it is \( 2\sqrt{h} \) meters. Its width is \( \Delta h \) meters, so its area is \( 2\sqrt{h} \Delta h \) square meters. Therefore the force on it is \((9 - h)(1000)(8)2\sqrt{h} \Delta h\) kilograms \cdot meters/second\(^2\). The total force is therefore

\[
\int_0^9 (9 - h)(1000)(8)2\sqrt{h} \, dh = \int_0^9 16000(9 - h)\sqrt{h} \, dh
\]

(in kilograms \cdot meters/second\(^2\)).

8. (12 points.) A 60 pound pail of water is attached to a rope, and is lifted from a depth of 100 feet in a well. The rope weighs \( \frac{1}{10} \) pounds per foot. How much work is required to lift the pail and the rope?

Solution: The pail weighs 60 pounds and is lifted 100 feet, so the work to lift the pail is 6000 foot-pounds.
For the rope, consider a short segment, with length $\Delta h$ meters, and located $h$ meters below the surface. It weighs $\frac{1}{10} \Delta h$ pounds, and it lifted $h$ feet, so the work to lift it is $\frac{1}{10} h \Delta h$ foot-pounds. So the work to lift the rope is, in foot-pounds,

$$\int_0^{100} \frac{h}{10} \, dh = \frac{h^2}{20} \bigg|_0^{100} = \frac{10000}{20} = 500.$$ 

So the total work is $6000 + 500 = 6500$ foot-pounds. (The units are required.)

9. (15 points.) Determine whether or not the improper integral $\int_{7}^{\infty} \frac{13 + 3 \sin(6e^y)}{y^5} \, dy$ converges.

Solution: For all real $y$ we have $-3 \leq 3 \sin(6e^y) \leq 3$, so $10 \leq 13 + 3 \sin(6e^y) \leq 16$. For $y > 0$ we therefore have

$$\frac{10}{y^5} \leq \frac{13 + 3 \sin(6e^y)}{y^5} \leq \frac{16}{y^5}.$$ 

In particular,

$$0 \leq \frac{13 + 3 \sin(6e^y)}{y^5} \leq \frac{16}{y^5}.$$ 

(You do need to check that the integrand in the problem is nonnegative.)

Now

$$\int_{7}^{\infty} \frac{16}{y^5} \, dy = -\frac{4}{y^4} \bigg|_{7}^{\infty} = \lim_{t \to \infty} \left( -\frac{4}{t^4} + \frac{4}{7^4} \right) = \frac{4}{7^4}.$$ 

So $\int_{7}^{\infty} \frac{16}{y^5} \, dy$ converges. Therefore the Comparison Test shows that $\int_{7}^{\infty} \frac{13 + 3 \sin(6e^y)}{y^5} \, dy$ converges.

10. (15 points.) Find the area of the region bounded by the curves $y = 2x^2$, $y = 3 - x$, and the $x$-axis, with $x \geq 0$. (It may help to draw a picture of the region first.)

Solution: Here is a picture of the region:
We will use vertical slices. We need the $x$-coordinates of the points where the curves intersect. We must solve three sets of simultaneous equations:

1. $y = 2x^2$ and $y = 0$, 
2. $y = 3 - x$ and $y = 0$, 
and
3. $y = 2x^2$ and $y = 3 - x$.

The only solution to (1) has $x = 0$, and the only solution to (2) has $x = 3$. The equations (3) imply $2x^2 + x - 3 = 0$, which has the solutions $x = 1$ and $x = -\frac{3}{2}$. We ignore $x = -\frac{3}{2}$ since we are supposed to assume that $x \geq 0$.

We need two integrals because the area consists of two regions bounded by different upper curves for $x \leq 1$ and $x \geq 1$. Thus the area is

$$\int_0^1 (2x^2 - 0) \, dx + \int_1^3 ((3 - x) - 0) \, dx = \int_0^1 2x^2 \, dx + \int_1^3 (3 - x) \, dx$$

$$= \frac{2x^3}{3} \bigg|_0^1 + \left(3x - \frac{x^2}{2}\right) \bigg|_1^3$$

$$= \frac{2}{3} + \left(3 \cdot 3 - \frac{3^2}{2} - 3 \cdot 1 + \frac{1^2}{2}\right) = \frac{8}{3}.$$

Alternate solution (sketch): We use the same picture as in the first solution, but we use horizontal slices, that is, we integrate with respect to $y$. The same computation as before also gives the $y$ coordinates of the intersection points, which are 0 for (1) and (3), and 2 for (2). We also have to rewrite all of the equations to give $x$ in terms of $y$. However, there is now only one integral, namely

$$\int_0^2 \left((3 - y) - \sqrt{y/2}\right) \, dy.$$

11. (15 points.) A solid has a base which is the disk in the $xy$ plane whose boundary is the circle $x^2 + y^2 = 9$. Its cross sections perpendicular to the the $x$-axis are squares. Find the volume of this solid.

Solution: The circle intersects the $x$-axis at $x = -3$ and $x = 3$. For $-3 \leq x \leq 3$, consider the cross section of the solid perpendicular to the $x$-axis. It has base length $2\sqrt{9 - x^2}$, so it has area

$$\left(2\sqrt{9 - x^2}\right)^2 = 4(9 - x^2).$$

If a cross section at this position has thickness $\Delta x$, and $\Delta x$ is small, then it has volume approximately $4(9 - x^2) \Delta x$. 

Now consider a subdivision of \([-3,3]\) into \(n\) intervals, each of length \(\Delta x = 6/n\). (We don’t have to use equal length intervals; this is just for convenience.) For \(k = 1,2,\ldots,n\) let \(x_k\) be a point in the \(k\)-th interval. Then the volume is approximately

\[
\sum_{k=1}^{n} 4(9 - x^2) \Delta x.
\]

This looks like a Riemann sum for

\[
\int_{-3}^{3} 4(9 - x^2) \, dx.
\]

So the volume is

\[
\int_{-3}^{3} 4(9 - x^2) \, dx = \int_{-3}^{3} (36 - 4x^2) \, dx = \left[ 36x - \frac{4x^3}{3} \right]_{-3}^{3} = 108 - 36 - (-108 + 36) = 144.
\]

(To get full credit, a solution does not have to contain an explicit Riemann sum as in (4) above. But the reasoning behind the expression

\[
\sum_{k=1}^{n} 4(9 - x^2) \Delta x \quad \text{or} \quad \int_{-3}^{3} 4(9 - x^2) \, dx
\]

must be made clear.)

12. (10 points.) You want to find \(\int (t^2 - 16)^{-5/2} \, dt\), for \(t > 4\). Using a suitable substitution, convert this indefinite integral into one involving trigonometric functions but only integer powers. Simplify the resulting integrand, but do not attempt to find the resulting integral.

**Solution:** Use the substitution \(t = 4 \sec(\theta)\), with \(0 < \theta < \frac{\pi}{2}\). Thus, \(dt = 4 \sec(\theta) \tan(\theta) \, d\theta\), so

\[
\int (t^2 - 16)^{-5/2} \, dt = \int (16 \sec^2(\theta) - 16)^{-5/2} \cdot 4 \sec(\theta) \tan(\theta) \, d\theta = \int 4^{-3/2} \sec(\theta) \tan(\theta) (\sec^2(\theta))^{-5/2} \, d\theta.
\]

We have \(\tan(\theta) > 0\) since \(0 < \theta < \frac{\pi}{2}\), and \(4^{-3/2} = \frac{1}{8}\), so, using the definitions of \(\tan(\theta)\) and \(\sec(\theta)\) at the last step,

\[
\int 4^{-3/2} \sec(\theta) \tan(\theta) (\sec^2(\theta))^{-5/2} \, d\theta = \int \frac{1}{8} \sec(\theta) \tan(\theta) (\sec(\theta))^{-5} \, d\theta
\]

\[
= \int \left( \frac{1}{8} \frac{\tan(\theta)}{\sec^4(\theta)} \right) d\theta = \int \frac{1}{8} \sin(\theta) \cos^3(\theta) \, d\theta.
\]

The last simplification is required.