1. Write as a single fraction, and simplify as much as possible: \( \frac{1}{x + 3} - \frac{1}{x - 7} \)

Solution:

\[
\frac{1}{x + 3} - \frac{1}{x - 7} = \frac{x - 7}{(x + 3)(x - 7)} - \frac{x + 3}{(x + 3)(x - 7)} = \frac{x - 7 - x - 3}{(x + 3)(x - 7)} = -\frac{10}{(x + 3)(x - 7)}.
\]

2. Let \( y(x) = \tan (x^5 + k \ln(x)) \), for a fixed constant \( k \). Find \( y'(x) \).

Solution: We use the chain rule, getting

\[
y'(x) = \sec^2 (x^5 + k \ln(x)) \cdot \frac{d}{dx} (x^5 + k \ln(x)) = \sec^2 (x^5 + k \ln(x)) \left( 5x^4 + \frac{k}{x} \right).
\]

3. Write as a single fraction, and simplify as much as possible: \( \frac{2}{q + 1} - \frac{1}{q + 4} \)

Solution:

\[
\frac{2}{q + 1} - \frac{1}{q + 4} = \frac{2(q + 4)}{(q + 1)(q + 4)} - \frac{q + 1}{(q + 1)(q + 4)} = \frac{2q + 8 - q - 1}{(q + 1)(q + 4)} = \frac{q + 7}{(q + 1)(q + 4)}.
\]

4. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”: \( \frac{2x^2 + y}{2x^2 + 3y} \)

Solution: The expression \( \frac{2x^2 + y}{2x^2 + 3y} \) can’t be simplified.

5. Find all real solutions to the equation \( 3(3x^{-2} + 2) = 6 \). If no real solution exists, write “no solution”.

Solution:

\[
9x^{-2} + 6 = 6
\]

\[
x^{-2} = 0
\]

Multiply both sides by \( x^2 \) to get \( 9 = 0 \). Therefore there are no solutions. (Alternatively, it is obvious that \( 9x^{-2} \) can never be zero.)

6. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”: \( \frac{2e^x + 5}{2e^x + 10} \)
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Solution: The expression \(\frac{2e^x + 5}{2e^x + 10}\) can’t be simplified.

7. Find all real solutions to the equation \(\frac{1}{25x^2} = 0\). If no real solution exists, write “no solution”.

Solution: Multiply both sides by \(25x^2\) to get \(1 = 0\). Therefore there are no solutions.
(Alternatively, it is obvious that \(\frac{1}{25x^2}\) can never be zero.)

8. Let \(g(x) = 3 - x^2\). Evaluate the expression \(g(-x) - g(x)\), and simplify it as much as possible.

Solution:
\[
g(-x) - g(x) = 3 - (-x)^2 - (3 - x^2) = 3 - x^2 - (3 - x^2) = 0.
\]

9. Let \(f(x) = 7 - x\). Evaluate the expression \(f(3) - f(2 - x)\), and simplify it as much as possible.

Solution:
\[
f(3) - f(2 - x) = 7 - 3 - (7 - (2 - x)) = 7 - 3 - 7 + 2 - x = -x - 1.
\]

10. Let \(f(t) = at^5 - \sec(t)\ln(t)\), for a fixed constant \(a\). Find \(f'(t)\).

Solution: We use the product rule on the second term, remembering to distribute the minus sign correctly. Thus
\[
f'(t) = a \cdot 5t^4 - \left[\sec(t)\tan(t)\ln(t) + \sec(t) \cdot \frac{1}{t}\right] = 5at^4 - \sec(t)\tan(t)\ln(t) - \sec(t) \cdot \frac{1}{t}.
\]

11. Find all real solutions to the equation \(\frac{20}{5x^2} = 0\). If no real solution exists, write “no solution”.

Solution: Multiply both sides by \(5x^2\) to get \(20 = 0\). Therefore there are no solutions.
(Alternatively, it is obvious that \(\frac{20}{5x^2}\) can never be zero.)

12. Multiply out: \((y - 5)(y^2 + 3y - 2)\).

Solution:
\[
(y - 5)(y^2 + 3y - 2) = y^3 - 5y^2 + 3y^2 - 15y - 2y + 10 = y^3 - 2y^2 - 17y + 10.
\]

13. Find all real numbers \(b\) such that \((-7, -b)\) is in the second quadrant (and not on any of the coordinate axes).

Solution: \((-7, -b)\) is in the second quadrant if and only if \(-b > 0\), which happens if and only if \(b < 0\).
14. Let \( f(t) = \frac{t^2}{e^t + ct} \), for a fixed constant \( c \). Find \( f'(t) \).

Solution:

\[
f'(t) = \frac{2t \cdot (e^t + t) - t^2 (e^t + c)}{(e^t + ct)^2} = \frac{2te^t + 2t^2 - t^2 e^t - ct^2}{(e^t + ct)^2} = \frac{2te^t - t^2 e^t + (2 - c)t^2}{(e^t + ct)^2}.
\]

The first step is the quotient rule, and the remaining steps are simplification.

15. Find all real solutions to the equation \( \frac{6}{7} + \frac{7}{x^2} = 1 \). If no real solution exists, write “no solution”.

Solution:

\[
\begin{align*}
6 + \frac{7}{x^2} &= 1 \\
\frac{7}{x^2} &= 1 - \frac{6}{7} = \frac{1}{7} \\
x^2 &= \frac{7}{1} \\
x^2 &= 7
\end{align*}
\]

Note that both answers actually are solutions to the original equation, that is, that multiplying both sides by \( x^2 \) at the second step did not introduce any extraneous solutions.

Since there is no partial credit, no credit is given for only one of the two solutions.

16. Find all real solutions to the equation \( \frac{6}{x + 5} + x = 0 \). If no real solution exists, write “no solution”.

Solution:

\[
\begin{align*}
\frac{6}{x + 5} + x &= 0 \\
6 + x(x + 5) &= 0 \\
x^2 + 5x + 6 &= 0 \\
(x + 2)(x + 3) &= 0 \\
x &= -3 \quad \text{or} \quad x &= -2.
\end{align*}
\]

Note that both answers actually are solutions to the original equation, that is, that multiplying both sides by \( x + 5 \) at the first step did not introduce any extraneous solutions.

Since there is no partial credit, no credit is given for only one of the two solutions.

17. Multiply out: \( (b - 2)(b^2 - 3b - 2) \).

Solution:

\[
(b - 2)(b^2 - 3b - 2) = b^3 - 2b^2 - 3b^2 + 6b - 2b + 4 = b^3 - 5b^2 + 4b + 4.
\]

18. Let \( g(t) = (2t^4 - q \arctan(t))^7 \), for a fixed constant \( q \). Find \( g'(t) \).
Solution: We use the chain rule, getting
\[
g'(t) = 7(2t^4 - q \arctan(t))^6 \cdot \frac{d}{dt}(2t^4 - q \arctan(t))
\]
\[
= 7(2t^4 - q \arctan(t))^6 \left(2 \cdot 4t^3 - \frac{q}{1 + t^2}\right) = 7(2t^4 - q \arctan(t))^6 \left(8t^3 - \frac{q}{1 + t^2}\right).
\]
The simplification in the last step is required.

19. Find all real solutions to the equation \(3\left[(x - 2)^{-2} + 2\right] = 6\). If no real solution exists, write “no solution”.

Solution:
\[
\begin{align*}
3\left[(x - 2)^{-2} + 2\right] &= 6 \\
(x - 2)^{-2} + 2 &= 2 \\
(x - 2)^{-2} &= 0
\end{align*}
\]
Multiply both sides by \((x - 2)^2\) to get \(1 = 0\). Therefore there are no solutions. (Alternatively, it is obvious that \((x - 2)^{-2} = \frac{1}{(x - 2)^2}\) can never be zero.)

20. Write as a single fraction, and simplify as much as possible: \(\frac{3}{y - 1} - \frac{1}{y - 2}\)

Solution:
\[
\frac{3}{y - 1} - \frac{1}{y - 2} = \frac{3(y - 2)}{(y - 1)(y - 2)} - \frac{y - 1}{(y - 1)(y - 2)} = \frac{3y - 6 - y + 1}{(y - 1)(y - 2)} = \frac{2y - 5}{(y - 1)(y - 2)}.
\]

21. Let \(f(t) = \frac{\cos(t)}{nt^2 + t}\), for a fixed constant \(n\). Find \(f'(t)\).

Solution:
\[
f'(t) = -\sin(t) \cdot \frac{(nt^2 + t) - \cos(t) \cdot (2nt + 1)}{(nt^2 + t)^2} = -\frac{(nt^2 + t) \sin(t) + (2nt + 1) \cos(t)}{(nt^2 + t)^2}
\]
by the quotient rule.

22. Let \(h(y) = \sec\left(y^5 - p \arcsin(y)\right)\), for a fixed constant \(p\). Find \(h'(y)\).

Solution: We use the chain rule, getting
\[
h'(y) = \sec \left(y^5 - p \arcsin(y)\right) \tan \left(y^5 - p \arcsin(y)\right) \cdot \frac{d}{dy} \left(y^5 - p \arcsin(y)\right)
\]
\[
= \sec \left(y^5 - p \arcsin(y)\right) \tan \left(y^5 - p \arcsin(y)\right) \left(5y^4 - \frac{p}{\sqrt{1 - y^2}}\right).
\]

23. Write as a single fraction, and simplify as much as possible: \(\frac{6}{c - 4} - \frac{1}{c - 2}\)

Solution:
\[
\frac{6}{c - 4} - \frac{1}{c - 2} = \frac{6(c - 2)}{(c - 4)(c - 2)} - \frac{c - 4}{(c - 4)(c - 2)} = \frac{6c - 12 - c + 4}{(c - 4)(c - 2)} = \frac{5c - 8}{(c - 4)(c - 2)}.
\]
24. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”:

\[
\frac{2y^2 + 2x}{2y^2 + 6x}
\]

Solution:

\[
\frac{2y^2 + 2x}{2y^2 + 6x} = \frac{2(y^2 + x)}{2(y^2 + 3x)} = \frac{y^2 + x}{y^2 + 3x}.
\]

The last expression can’t be further simplified.

25. Multiply out: \((b + 5)(b^2 - 2b + 3)\).

Solution:

\[
(b + 5)(b^2 - 2b + 3) = b^3 + 5b^2 - 2b^2 - 10b + 3b + 15 = b^3 + 3b^2 - 7b + 15.
\]

26. Let \(g(x) = 21x^4 - b \cos(x) \arctan(x)\), for a fixed constant \(b\). Find \(g'(x)\).

Solution: We use the product rule on the second term, remembering to distribute the minus sign correctly. Thus

\[
g'(x) = 21\cdot 4x^3 - b \left[ -\sin(x) \arctan(x) + \cos(x) \cdot \frac{1}{1 + x^2} \right] = 84x^3 + b \frac{\sin(x) \arctan(x)}{1 + x^2} - b \cos(x) \frac{1}{1 + x^2}.
\]

27. Let \(f(t) = 10[\ln(t)]^6 - pe^t\), for a fixed constant \(p\). Find \(f'(t)\).

Solution: We use the chain rule on the first term, getting

\[
f'(t) = 10 \cdot 6[\ln(t)]^5 \cdot \frac{1}{t} - pe^t = \frac{60[\ln(t)]^5}{t} - pe^t.
\]

28. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”:

\[
\frac{3 \cos(5z)}{3 \cos(5z) - 6}
\]

Solution:

\[
\frac{3 \cos(5z)}{3 \cos(5z) - 6} = \frac{3[\cos(5z)]}{3[\cos(5z) - 2]} = \frac{\cos(5z)}{\cos(5z) - 2}.
\]

The last expression can’t be further simplified.

29. Let \(y(x) = (ax^7 - \tan(x))^3\), for a fixed constant \(a\). Find \(y'(x)\).

Solution: We use the chain rule, getting

\[
y'(x) = 3(ax^7 - \tan(x))^2 \cdot \frac{d}{dx}(ax^7 - \tan(x))
\]

\[
= 3(ax^7 - \tan(x))^2 (a \cdot 7x^6 - \sec^2(x)) = 3(ax^7 - \tan(x))^2 (7ax^6 - \sec^2(x)).
\]

The simplification in the last step is required.

30. Find all real solutions to the equation \(2 \left( \frac{1}{x + 3} \right) = 6\). If no real solution exists, write “no solution”.


Solution:
\[
2 \left( \frac{1}{x + 7} + 3 \right) = 6
\]
\[
\frac{2}{x + 7} + 6 = 6
\]
\[
\frac{2}{x + 7} = 0
\]
Multiply both sides by \(x + 7\) to get \(2 = 0\). Therefore there are no solutions. (Alternatively, \(\frac{2}{x + 7}\) can obviously never be zero.)

31. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”: \(\frac{4e^x}{4e^x + 6}\)

Solution:
\[
\frac{4e^x}{4e^x + 6} = \frac{2(2e^x)}{2(2e^x + 3)} = \frac{2e^x}{2e^x + 3}
\]
The last expression can’t be further simplified.

32. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”: \(\frac{2w^2 - 2}{2w^2 - 6}\)

Solution:
\[
\frac{2w^2 - 2}{2w^2 - 6} = \frac{2(w^2 - 1)}{2(w^2 - 3)} = \frac{w^2 - 1}{w^2 - 3}
\]
The last expression can’t be further simplified.

33. Write as a single fraction, and simplify as much as possible: \(\frac{3}{w + 1} - \frac{1}{w - 5}\)

Solution:
\[
\frac{3}{w + 1} - \frac{1}{w - 5} = \frac{3(w - 5) - (w + 1)}{(w + 1)(w - 5)} = \frac{3w - 15 - w - 1}{(w + 1)(w - 5)} = \frac{2w - 16}{(w + 1)(w - 5)}.
\]

34. Let \(g(x) = 3 - x^2\). Evaluate the expression \(g(1 - x) - g(x)\), and simplify it as much as possible.

Solution:
\[
g(1 - x) - g(x) = 3 - (1 - x)^2 - (3 - x^2)
\]
\[
= 3 - (1 - 2x + x^2) - (3 - x^2) = 3 - 1 + 2x - x^2 - 3 + x^2 = 2x - 1.
\]

35. Which of the following four expressions can represent a strictly positive number for some real value of \(x\)? List all possibilities: \(-x, -x^2, -|x|, |x|\).

Solution: The expression \(-x^2\) is never strictly positive, since \(x^2 \geq 0\) for all real \(x\). Similarly, the expression \(-|x|\) is never strictly positive, since \(|x| \geq 0\) for all real \(x\). However, the expression \(-x\) can be strictly positive. (For example, take \(x = -3\)). Also, the
expression \(|-x|\) can be strictly positive—this happens for every nonzero real number \(x\), in particular, for both \(x = 7\) and \(x = -7\). The answer is therefore: \(-x\) and \(|-x|\).

36. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”: \(\frac{\sin(7x) + 7}{\sin(7x) - 7}\)

Solution: The expression \(\frac{\sin(7x) + 7}{\sin(7x) - 7}\) can’t be simplified.

37. Let \(f(x) = \tan(kx^6 + 2x^4 + 37)\), for a fixed constant \(k\). Find \(f'(x)\).

Solution: We use the chain rule, getting

\[f'(x) = \sec^2(kx^6 + 2x^4 + 37) \cdot (6kx^5 + 8x^3) = (6kx^5 + 8x^3) \sec^2(kx^6 + 2x^4 + 37).\]

38. Let \(f(y) = 2\cos^{12}(y) + d\ln(y)\), for a fixed constant \(d\). Find \(f'(y)\).

Solution: We use the chain rule on the first term, getting

\[f'(y) = 2 \cdot 13 \cos^{12}(y)(-\sin(y)) + d \cdot \frac{1}{y} = -26 \cos^{12}(y) \sin(y) + \frac{d}{y}.\]

39. Find all real numbers \(y\) such that \(3 + y^2 \leq 17\).

Solution:

\[
3 + y^2 \leq 17 \\
y^2 \leq 14 \\
-\sqrt{14} \leq y \leq \sqrt{14}.
\]

The second step was done by subtracting 3 everywhere.

40. Write as a single fraction, and simplify as much as possible: \(\frac{2}{c + 6} - \frac{1}{c + 2}\)

Solution:

\[
\frac{2}{c + 6} - \frac{1}{c + 2} = \frac{(c + 2) - 6}{(c + 6)(c + 2)} = \frac{c + 4 - 6}{(c + 6)(c + 2)} = \frac{c}{(c + 6)(c + 2)}.
\]

41. Find all real numbers \(x\) such that \(|-x| = -x\).

Solution: We know that \(|-x| = |x|\) for all real \(x\). Thus,

\[
|x| = \begin{cases} 
    x & \text{if } x \geq 0 \\
    -x & \text{if } x < 0.
\end{cases}
\]

If \(x > 0\) then \(x \neq -x\), so \(|-x| \neq -x\). If \(x = 0\), then \(|-x|\) and \(-x\) are both zero, so \(|-x| = -x\). If \(x < 0\), then \(|-x| = -x\) by the formula above. Thus \(|-x| = |x|\) exactly when \(x \leq 0\).
42. Simplify the following expression as much as possible. If no simplification is possible, write “not possible”:
\[
\frac{xe^x + x^2}{2xe^x + x^2}
\]
Solution:
\[
\frac{xe^x + x^2}{2xe^x + x^2} = \frac{x(e^x + x)}{x(2e^x + x)} = \frac{e^x + x}{2e^x + x}.
\]
The last expression can’t be further simplified.

43. Find all real solutions to the equation \(7 - 8x^{-1} = x\). If no real solution exists, write “no solution”.

Solution:
\[7 - 8x^{-1} = x\]
Multiply both sides by \(x\), getting:
\[7x - 8 = x^2\]
\[x^2 - 7x + 8 = 0\]
\[x = \frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 8}}{2} = \frac{7 \pm \sqrt{17}}{2}\]
Note that both answers actually are solutions to the original equation, that is, that multiplying both sides by \(x\) at the first step did not introduce any extraneous solutions.
Since there is no partial credit, no credit is given for only one of the two solutions.