Submit with your exam the statement (presumably handwritten), with your signature, “I understand the instructions on academic integrity for this exam, and will follow them. I understand that being found responsible for academic misconduct is considered cheating and will result in a score of zero on this exam.”

GENERAL INSTRUCTIONS (The problems are on the next page.)

(1) The point values are as indicated; total 100 points (not counting extra credit).
(2) This exam is open book and open notes. All work must be your own; you may not collaborate with others. This means that no interactive help (in person, internet based, or any other method) is allowed. A graphing calculator (not on your computer) is allowed, but computer algebra systems are not. All electronic devices other than your computer, cell phone (if needed for scanning), and calculator must be off and out of reach; the cell phone must be visible but off when not being used for scanning. Normal rules on academic integrity apply.
(3) Starting time 26 February 2021 at 8:00 am Pacific Standard Time. Your work must be submitted by 9:05 am Pacific Standard Time. For a nominally 50 minute exam, this allows 5 minutes for downloading at the beginning and 10 minute for scanning etc. at the end. You must stay until 9:00 am, even if you are done. Use the time to check answers, or read a book.
(4) Show enough of your work that your method is obvious. Be sure that every statement you write is correct and uses correct notation. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
(5) Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
(6) When exact values are specified, give answers such as \( \frac{1}{7}, \sqrt{2}, \ln(2) \), or \( \frac{2\pi}{9} \). Calculator approximations will not be accepted.
(7) Final answers must always be simplified unless otherwise specified.

EXCERPTS FROM THE INSTRUCTIONS FOR REMOTE TESTING (See the separate sheet for the full set of instructions.)

(1) You must be present in the normal class Zoom meeting (ID 916 3410 5223) while taking the exam. Zoom settings:
   - Virtual background off.
   - Video and web camera on, and set to show you and your workspace.
   - Microphone on.
   - Speaker on (to hear announcements).
(2) Submit questions to the instructor via chat. They will be answered via chat, unless they are something the whole class should know about, in which case they will be answered via a spoken announcement.
(3) Alternate submission, in case you have trouble with Canvas: by email to ncp@uoregon.edu. The file must be contained in the email message; files stored on Google Docs or similar sites will not be retrieved. (I will check the time stamps on email.)

Date: 26 February 2021.
1. (1 point.) Integration by trigonometric substitutions was invented for the purpose of torturing which of:
   
   (1) Functions whose definitions contain square roots.
   (2) Calculus students.
   (3) Professors who have to teach it in calculus classes.

   Solution: I think the answer is (3).

2. (12 points.) Find the area of the region bounded by the curves \( y = e^x \), \( y = 2e^x \), \( y = 1 \), and \( y = 3 \). (It may help to draw a picture of the region first.)

   Solution: Here is a picture of the region:

   ![Picture of the region]

   It is much easier to find the area using horizontal slices (that is, integrating with respect to \( y \)). So we write the equations of the curves as

   \[ x = \ln(y) \quad \text{and} \quad x = \ln\left(\frac{y}{2}\right) = \ln(y) - \ln(2). \]

   Then the area is

   \[
   \int_1^3 \left[ \ln(y) - \ln\left(\frac{y}{2}\right) \right] dy = \int_1^3 \left[ \ln(y) - \left( \ln(y) - \ln(2) \right) \right] dy = \int_1^3 \ln(2) \ dy = 2\ln(2).
   \]

3. (12 points.) The market value of farmland in the valley of the Nograbore River along a long straight section of the river is modelled as 300 \(- 2x\) million dollars per mile along the river, where \( x \) is measured in miles from the mouth of the river, and \( 0 \leq x \leq 100 \). Set up an integral which represents the total market value of land in the valley of the Nograbore River between 20 and 50 miles from its mouth.

   Solution: Consider the section of the river between \( x \) and \( x + \Delta x \). The market value of the farmland is approximately 300 \(- 2x\) million dollars per mile in this interval, provided \( \Delta x \) is small.
Now consider a subdivision of $[20, 50]$ into $n$ intervals, each of length $\Delta x = 30/n$. (We don’t have to use equal length intervals; this is just for convenience.) For $k = 1, 2, \ldots, n$ let $x_k$ be the left endpoint of the $k$-th interval. Then the total market value of the farmland along the Nograbore River between 20 and 50 miles from its mouth is approximately

\[
\sum_{k=1}^{n} (300 - 2x_k) \Delta x.
\]

This is a Riemann sum for $\int_{20}^{50} (300 - 2x) \, dx$. Therefore, according to the model, the actual total market value of the farmland along the Nograbore River between 20 and 50 miles from its mouth is

\[
\int_{20}^{50} (300 - 2x) \, dx.
\]

(To get full credit, a solution does not have to contain an explicit Riemann sum as in (1) above. But the reasoning behind the expression $(300 - 2x_k) \Delta x$ or $(300 - 2x) \, dx$ must be made clear.)

4. (12 points.) Define

\[
h(x) = (x^3 - 2x^2 + 9) \int_{-15}^{x} s \sin^3(e^s) \, ds.
\]

Find $h'(x)$. (There might be an integral in your answer.)

**Solution:** Define

\[
F(x) = \int_{-15}^{x} s \sin^3(e^s) \, ds.
\]

Then (using the Fundamental Theorem of Calculus for the second formula)

\[
h(x) = (x^3 - 2x^2 + 9)F(x) \quad \text{and} \quad F'(x) = x \sin^3(e^x).
\]

Using the product rule, we get

\[
h'(x) = \frac{d}{dx} (x^3 - 2x^2 + 9)F(x) + (x^3 - 2x^2 + 9)F'(x)
\]

\[
= (3x^2 - 4x) \int_{-15}^{x} s \sin^3(e^s) \, ds + x(x^3 - 2x^2 + 9) \sin^3(e^x).
\]

(The variable $s$ may not appear anywhere in the answer except inside the integral in the first term, because $h'(x)$ must be a function of $x$.)

5. (12 points.) Consider the region between the curve $y = x^{-3/4}$ and the $x$-axis, and to the right of the line $x = 4$. It is rotated about the $x$-axis. Find its volume (possibly infinite).

**Solution:** Here are pictures: the graphs of the bounding curves, only going out to $x = 12$, on the left, and the solid on the right, again only going out to $x = 12$. 
4 SOLUTIONS TO MIDTERM 2 VERSION 1

We use vertical disks. Consider a position \( x \) along the \( x \)-axis, with \( x \geq 4 \), and a thin vertical slice of the solid, with small thickness \( \Delta x \). The corresponding thickened disk has radius \( x^{-3/4} \) and thickness \( \Delta x \), so has volume \( \pi (x^{-3/4})^2 \Delta x = \pi x^{-3/2} \Delta x \). Considering Riemann sums, we thus see that the volume of the part of the solid between \( x = 4 \) and \( x = t \) is

\[
\int_4^t \pi x^{-3/2} \, dx = \left[ -\pi \cdot 2x^{-1/2} \right]_4^t = -\frac{2\pi}{\sqrt{t}} + \frac{2\pi}{\sqrt{4}} = \pi - \frac{2\pi}{\sqrt{t}}.
\]

Therefore the volume of the solid is

\[
\lim_{t \to \infty} \left( \pi - \frac{2\pi}{\sqrt{t}} \right) = \pi - \lim_{t \to \infty} \frac{2\pi}{\sqrt{t}} = \pi.
\]

In particular, it is finite.

One can also write the solution as

\[
\int_4^\infty \pi (x^{-3/4})^2 \, dx = \int_4^\infty \pi x^{-3/2} \, dx = \left[ -\pi \cdot 2x^{-1/2} \right]_4^\infty = \lim_{t \to \infty} \left( -\frac{2\pi}{\sqrt{t}} + \frac{2\pi}{\sqrt{4}} \right) = \pi.
\]

6. (12 points.) Find \( \int \sin^3(6x) \, dx \).

**Solution:** We have

\[
\int \sin^3(6x) \, dx = \int \sin(6x) (1 - \cos^2(6x)) \, dx = \int \sin(6x) \, dx - \int \sin(6x) \cos^2(6x) \, dx.
\]

On the first integral, use the substitution \( u = 6x \), so \( dx = \frac{1}{6} \, du \), getting

\[
\int \sin(6x) \, dx = -\frac{1}{6} \cos(6x) + C_1.
\]

On the first integral, use the substitution \( u = \cos(6x) \), so \( \sin(6x) \, dx = -\frac{1}{6} \, du \), getting

\[
\int \sin(6x) \cos^2(6x) \, dx = -\frac{1}{18} \cos^3(6x) + C_2.
\]

So (combining the constants of integration)

\[
\int \sin^3(6x) \, dx = -\frac{1}{6} \cos(6x) + \frac{1}{18} \cos^3(6x) + C.
\]

7. (13 points.) Let \( F \) be a function such that \( F'(t) = \sin(t^4) \) for all real \( t \). Find

\[
\int t^2 F(t) \, dt
\]
in terms of elementary functions and $F$.

Solution: Integrate by parts, taking 

$$u(t) = F(t), \quad v'(t) = t^2, \quad u'(t) = \sin(t^4), \quad \text{and} \quad v(t) = \frac{1}{3}t^3.$$ 

Thus,

$$\int t^2 F(t) \, dt = \frac{1}{3} t^3 F(t) - \int \frac{1}{3} t^3 \sin(t^4) \, dt.$$ 

We do the remaining integral using the substitution $w = t^4$. Thus, $dw = -4t^3 \, dt$, so

$$t^3 \, dt = \frac{-1}{4} dw,$$ 

and

$$\int \frac{1}{3} t^3 \sin(t^4) \, dt = \frac{-1}{12} \int \sin(w) \, dw = \frac{-1}{12} \cos(w) + C_0 = \frac{-\cos(t^4)}{12} + C_0.$$ 

So

$$\int t^2 F(t) \, dt = \frac{1}{3} t^3 F(t) - \int \frac{1}{3} t^3 \sin(t^4) \, dt = \frac{1}{3} t^3 F(t) + \frac{\cos(t^4)}{12} - C_0 = \frac{1}{3} t^3 F(t) + \frac{\cos(t^4)}{12} + C,$$

(with $C = -C_0$). Just writing the last line shows enough work.

8. (13 points.) Determine whether or not the integral $\int_0^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx$ converges.

Solution: We have

$$\int_0^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx = \int_0^1 \frac{1}{1 + \frac{1}{2}x^4} \, dx + \int_1^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx.$$ 

Also $\int_0^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx$ is the integral of a continuous function on a closed bounded interval, so converges (it isn’t an improper integral). So it is enough to show that $\int_1^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx$ converges.

We do this by comparison with $f(x) = \frac{2}{x^2}$.

For $x \geq 1$, we have $x^4 \geq x^2$, so

$$1 + \frac{x^4}{2} \geq 1 + \frac{x^2}{2} \geq \frac{x^2}{2} > 0,$$

and

$$0 \leq \frac{1}{1 + \frac{1}{2}x^4} \leq \frac{1}{(\frac{1}{2}x^2)} = \frac{2}{x^2}.$$ 

Moreover,

$$\int_1^\infty \frac{2}{x^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{2}{x^2} \, dx = \lim_{b \to \infty} \left( -\frac{2}{x} \bigg|_1^b \right) = \lim_{b \to \infty} \left( -\frac{2}{b} + 2 \right) = 2,$$

so $\int_1^\infty \frac{2}{x^2} \, dx$ converges. Therefore the Comparison Test shows that $\int_0^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx$ converges.
Alternate solution: We use the Comparison test with the function \( g(x) = \frac{2}{1 + x^2} \). For \( x \geq 1 \), we have \( x^4 \geq x^2 \), so

\[
1 + \frac{x^4}{2} \geq 1 + \frac{x^2}{2} \geq \frac{1}{2} + \frac{x^2}{2} > 0,
\]
and

\[
0 \leq \frac{1}{1 + \frac{1}{2}x^4} \leq \frac{1}{\frac{1}{2} + \frac{x^2}{2}} = \frac{2}{1 + x^2}.
\]

Moreover,

\[
\int_0^\infty \frac{2}{1 + x^2} \, dx = \lim_{b \to \infty} \int_0^b \frac{2}{1 + x^2} \, dx = \lim_{b \to \infty} \left( 2 \arctan(x) \right|_0^b) = \lim_{b \to \infty} \left( 2 \arctan(x) - 2 \arctan(0) \right) = \pi,
\]
so \( \int_0^\infty \frac{2}{1 + x^2} \, dx \) converges. Therefore the Comparison Test shows that \( \int_0^\infty \frac{1}{1 + \frac{1}{2}x^4} \, dx \) converges.

9. (13 points.) A micrometeorite strikes a flat part of the surface of the sixth moon of the gas giant planet Groggxth. It creates a crater which is shaped like an upside down hemisphere with diameter 4 meters. How much work is required to remove the rock from this crater and spread it out as a thin layer of dust on the surrounding flat area? Ignore the thickness of the dust layer? The rock has a density of 4000 kilograms per cubic meter, and the acceleration due to gravity on the surface of the sixth moon of Groggxth is \( g = 3 \text{ m/sec}^2 \).

Solution: The crater has radius and depth 2 meters. For \( 0 \leq y \leq 2 \), consider a thin layer of rock at depth \( y \) meters, and with small thickness \( \Delta y \). Its shape is a thin circular disk, with radius \( \sqrt{4 - y^2} \) meters and thickness \( \Delta y \) meters, so its volume is \( \pi(4 - y^2)^2 \Delta y = \pi(4 - y^2) \Delta y \) cubic meters. Therefore its mass is \( m = 4000\pi(4 - y^2) \Delta y \) kilograms. It is lifted \( y \) meters against a force of \( mg = 3 \cdot 4000\pi(4 - y^2) \Delta y \) Newtons, so the work is \( mgy = 3 \cdot 4000\pi(4 - y^2)y \Delta y \) Newton-meters. By considering Riemann sums, we see that we need:

\[
\int_0^2 3 \cdot 4000\pi(4 - y^2)y \, dy = \int \left( 48,000\pi y - 12,000y^3 \right) \, dy = \left( 24,000\pi y^2 - 3000\pi y^4 \right|_0^2
\]

\[
= 24,000\pi \cdot 2^2 - 3000\pi \cdot 2^4 = 48,000\pi.
\]

The total work is thus 48,000\( \pi \) Newton-meters.

(The units are required.)

Extra credit. (15 extra credit points.) (Do not attempt this problem until you have finished all the ordinary problems on the exam and checked your answers to them. It problem will only be counted if you get at least 75 points on the main part of this exam.)
Let $f$ be the function $f(x) = x^3 + 2x + 3$ for all real numbers $x$. This function has an inverse $f^{-1}$, also defined for all real numbers. Find $\int_3^{15} f^{-1}(x) \, dx$. Your answer should be a real number; $f$ and $f^{-1}$ may not appear in it.

**Solution:** One can check (just by guessing) that $f(0) = 3$ and $f(2) = 15$. Therefore $f^{-1}(3) = 0$ and $f^{-1}(15) = 2$.

We do the integral using the substitution $u = f^{-1}(x)$. This gives $x = f(u)$, so $dx = f'(u) \, du = (3u^2 + 2) \, du$. Since $u = 0$ when $x = 3$ and $u = 2$ when $x = 15$, we get

$$\int_3^{15} f^{-1}(x) \, dx = \int_0^2 u(3u^2 + 2) \, du = \int_0^2 (3u^3 + 2u) \, du$$

$$= \left[ \frac{3u^4}{4} + u^2 \right]_0^2 = 12 + 4 - (0 + 0) = 16.$$

**Alternate solution:** We do the indefinite integral using the substitution $u = f^{-1}(x)$. This gives $x = f(u)$, so $dx = f'(u) \, du = (3u^2 + 2) \, du$. Therefore

$$\int f^{-1}(x) \, dx = \int u(3u^2 + 2) \, du = \int (3u^3 + 2u) \, du$$

$$= \frac{3u^4}{4} + u^2 + C = \frac{3[f^{-1}(x)]^4}{4} + [f^{-1}(x)]^2 + C.$$

One can check (just by guessing) that $f(0) = 3$ and $f(2) = 15$. Therefore $f^{-1}(3) = 0$ and $f^{-1}(15) = 2$. So

$$\int_3^{15} f^{-1}(x) \, dx = \left( \frac{3[f^{-1}(x)]^4}{4} + [f^{-1}(x)]^2 \right) \bigg|_3^{15} = \left( \frac{3 \cdot 2^4}{4} + 2^2 \right) - \left( \frac{3 \cdot 0^4}{4} + 0^2 \right) = 16.$$