Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Monday, this means complaints must be received by Tuesday 27 Oct.

1. (8 points.) Here is the graph of a function $y = f(x)$:

![Graph of a function](image)

Find $\int_1^6 f(x) \, dx$.

*Solution:* We interpret $\int_1^6 f(x) \, dx$ as the area between the graph and the $x$-axis, taking areas below the $x$-axis to be negative. Thus, using the formula for the area of a triangle for the first two terms,

$$\int_1^6 f(x) \, dx = \int_1^2 f(x) \, dx + \int_2^4 f(x) \, dx + \int_4^6 f(x) \, dx = \frac{1}{2} (1)(1) - \frac{1}{2} (2)(2) - (2)(2) = \frac{1}{2} - 2 - 4 = -\frac{11}{2}.$$

(One can also just count grid boxes and halves of grid boxes.)

2. (10 points.) Find $\int_0^4 \left( e^{-3q} - \frac{1}{1 + q^2} \right) dq$.

*Solution:* We have, using the substitution $u = -3q$ at the second step,

$$\int_0^4 \left( e^{-3q} - \frac{1}{1 + q^2} \right) dq = \int_0^3 e^{-3q} \, dq - \int_0^3 \frac{1}{1 + q^2} \, dq = \int_0^{-12} \left( -\frac{1}{3} \right) e^u \, du - (\arctan(q)) \bigg|_0^4$$

$$= \left( -\frac{1}{3} \right) e^u \bigg|_0^{-12} - (\arctan(4) - \arctan(0))$$

$$= -\frac{1}{3} e^{-12} + \frac{1}{3} e^0 - (\arctan(4) - \arctan(0))$$

$$= \frac{1}{3} - \frac{1}{3} e^{-12} - \arctan(4).$$

The simplifications $e^0 = 1$ and $\arctan(0) = 0$ are required.

3. (10 points.) Find $\int q \sin(q) \, dq$.

*Solution:* Integrate by parts with $u(q) = q$, $v'(q) = \sin(q)$, $u'(q) = 1$, and $v(q) = -\cos(q)$. Then

$$\int q \sin(q) \, dq = q(-\cos(q)) - \int (-\cos(q)) \, dq$$

$$= -q \cos(q) + \sin(q) + C.$$
4. (9 points.) Calculate and simplify the Riemann sum to approximate \( \int_{-1}^{5} (x^2 - 2x) \, dx \) using 3 equal length subintervals and left endpoints.

**Solution:** Set \( f(x) = x^2 - 3x \). The intervals have length \((5 - (-1))/3 = 2\), so they are 

\([-1, 1], \quad [1, 3], \quad \text{and} \quad [3, 5]\).

Therefore the Riemann sum is 

\[ f(-1) \cdot 2 + f(1) \cdot 2 + f(3) \cdot 2 = 3 \cdot 2 + (-1) \cdot 2 + 3 \cdot 2 = 10. \]

5. (10 points.) Find \( \int r \cos(7 + 4 \ln(r)) \, dr \).

**Solution:** Substitute \( u = 7 + 4 \ln(r) \), so that 

\[ \frac{du}{dr} = \frac{4}{r} \quad \text{and} \quad \frac{1}{r} \, dr = \frac{1}{4} \, du. \]

This gives 

\[ \int \frac{1}{r} \cos(7 + 4 \ln(r)) \, dr = \int \left(\frac{1}{4}\right) \cos(u) \, du = \frac{1}{4} \sin(u) + C = \frac{1}{4} \sin(7 + 4 \ln(r)) + C. \]

6. (12 points.) Let \( P \) be a function such that \( P'(x) = \cos(x^3) \) for all real \( x \). Find \( \int 3xP(x) \, dx \) in terms of elementary functions and \( P \).

**Solution:** Integrate by parts, taking 

\[ u(x) = P(x), \quad u'(x) = 3x, \quad u'(x) = \cos(x^3), \quad \text{and} \quad v(x) = \frac{3}{2} x^2. \]

Thus, 

\[ \int xP(x) \, dx = \frac{3}{2} x^2 P(x) - \int \frac{3}{2} x^2 \cos(x^3) \, dx. \]

We do the remaining integral using the substitution \( u = x^3 \). Thus, \( dw = 3x^2 \, dx \), so \( x^2 \, dx = \frac{1}{3} \, dw \), and 

\[ \int \frac{3}{2} x^2 \cos(x^3) \, dx = \int \left(\frac{3}{2}\right) \left(\frac{1}{3}\right) \cos(u) \, dw = \frac{1}{2} \sin(u) + C_0 = \frac{1}{2} \sin(x^3) + C_0. \]

So (with \( C = -C_0 \)) 

\[ \int xP(x) \, dx = \frac{3}{2} x^2 P(x) - \int \frac{3}{2} x^2 \cos(x^3) \, dx = \frac{3}{2} x^2 P(x) - \frac{1}{2} \sin(x^3) - C_0 = \frac{3}{2} x^2 P(x) - \frac{1}{2} \sin(x^3) + C. \]

Just writing the last line shows enough work.

7. (10 points.) Your math tutor claims that 

\[ \int x^4 e^{-x} \, dx = -(24 + 24x + 12x^2 + 4x^3 + x^4) e^{-x} + C. \]

Because this tutor has previously made mistakes, you are skeptical of this answer. Check whether this answer is right. Show your work.
8. (11 points.) Define \( h(y) = \int_{-10}^{2y^3} \sin(t^3 + 5t + 9) \, dt \). Find \( h'(y) \).

**Solution:** Define

\[
F(x) = \int_{-10}^{x} \sin(t^3 + 5t + 9) \, dt.
\]

(You may not define

\[
h(x) = \int_{-10}^{x} \sin(t^3 + 5t + 9) \, dt,
\]

since the letter \( h \) has already been used for a different function.) Then (using the Fundamental Theorem of Calculus for the second formula)

\[
h(y) = F(2y^3) \quad \text{and} \quad F'(x) = \sin(x^3 + 5x + 9).
\]

Using the chain rule, we get

\[
h'(z) = \frac{d}{dy} (F(2y^3)) = F'(2y) \frac{d}{dy} (2y^3) = \sin \left( (2y^3)^3 + 5(2y^3) + 9 \right) \cdot 6y^2 = 6y^2 \sin(8y^9 + 10y^3 + 9).
\]

(The variable \( t \) may not appear anywhere in the answer, because \( h'(y) \) must be a function of \( y \).)

9. (10 points.) Find an antiderivative \( F \) of the function \( f(x) = 3x^2 - 6x + 5 \) such that \( F(2) = -1 \).

**Solution:** We first need

\[
\int (3x^2 - 6x + 5) \, dx = x^3 - 3x^2 + 5x + C.
\]

Now we must choose \( C \) such that the function \( F(x) = x^3 - 3x^2 + 5x + C \) satisfies \( F(2) = -1 \). Thus,

\[
-1 = F(2) = 2^3 - 3 \cdot 2^2 + 5 \cdot 2 + C = 6 + C,
\]

so \( C = -7 \). Thus, we take

\[
F(x) = x^3 - 3x^2 + 5x - 7.
\]

10. (10 points.) Define

\[
h(t) = \begin{cases} 
6\sqrt{t} & 0 \leq t \leq 4 \\
\frac{20}{t^2} & t > 4.
\end{cases}
\]

Find \( \int_{1}^{5} h(t) \, dt \).

**Solution:** We can write

\[
\int_{1}^{5} h(t) \, dt = \int_{1}^{4} h(t) \, dt + \int_{4}^{5} h(t) \, dt = \int_{1}^{4} t^{1/2} \, dt + \int_{4}^{5} 20t^{-2} \, dt.
\]
We have
\[ \int_1^4 t^{1/2} \, dt = 6 \left( \frac{2}{3} \right) t^{3/2} \bigg|_1^4 = 4t^{3/2} \bigg|_1^4 = 4 \cdot 4^{3/2} - 4 \cdot 1^{3/2} = 4 \cdot 8 - 4 = 28 \]
and
\[ \int_4^5 20t^{-2} \, dt = -20t^{-1} \bigg|_4^5 = -\frac{20}{5} + \frac{20}{4} = 1. \]
Therefore
\[ \int_1^5 h(t) \, dt = 28 + 1 = 29. \]

**EC. (20 extra credit points)** Define
\[ w(x) = \int_3^x \sin(e^s) \, ds \] 
\[ \cos(t^5) \, dt. \]

Find \( w'(x). \) (There will be an integral sign in your answer.)

**Solution:** Define
\[ F(x) = \int_9^x \sin(e^s) \, ds \quad \text{and} \quad G(x) = \int_3^x \cos(t^5) \, dt. \]

Then (using the Fundamental Theorem of Calculus for the second and third formulas)
\[ w(x) = G(F(x^2)), \quad F'(x) = \sin(e^x), \quad \text{and} \quad G'(x) = \cos(x^5). \]

Using the chain rule twice, we get
\[ w'(x) = \frac{d}{dx} \left( G(F(x^2)) \right) = G'(F(x^2)) \frac{d}{dx} (F(x^2)) = \cos \left( (F(x^2))^5 \right) F'(x^2) \frac{d}{dx} (x^2) \]
\[ = \cos \left( \left[ \int_9^{x^2} \sin(e^s) \, ds \right]^5 \right) \sin \left( e^{x^2} \right) \cdot 2x = 2x \sin \left( e^{x^2} \right) \cos \left( \left[ \int_9^{x^2} \sin(e^s) \, ds \right]^5 \right). \]