Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Friday, this means complaints must be received by Monday 5 March.

1. (11 points.) A particle moves along a straight line in such a way that, \( t \) hours after it starts moving, its distance to the right of its starting point is \( 3t^2 + 4(8t + 1)^{-1/2} \) centimeters. During the period from 1 hour to 3 hours after its starting time, what is the average distance of the particle to the right of its starting point?

**Solution:** Let \( p(t) \) be the distance to the right of the starting point at time \( t \), in centimeters. We need to find the average value of \( p(t) \) for \( t \) in \([1, 3]\), which is (at the second step, for the second integral using the substitution \( u = 8t + 1, \) so \( dt = \frac{5}{8} \, du \))

\[
\frac{1}{3-1} \int_1^3 p(t) \, dt = \frac{1}{2} \left( \int_1^3 3t^2 \, dt + \int_1^3 4(8t + 1)^{-1/2} \, dt \right)
\]

\[
= \frac{1}{2} \left( t^3 \bigg|_1^3 + 4 \left( \frac{1}{1/2} \right) \left( \frac{1}{8} \right) (8t + 1)^{1/2} \bigg|_1^3 \right) = \frac{1}{2} (3^3 + \sqrt{25}) - \frac{1}{2} (1^3 + \sqrt{9}) = 14.
\]

So the average distance to the right of the starting point is 14 centimeters.

(The units in the final answer are **required**.)

2. (9 points.) Write down a formula in terms of integrals for the area of the region bounded by the curves \( y = x^2, \ y = (x - 4)^2, \) and \( y = (x - 2)^2 - 4. \) (It may help to draw a picture of the region first.) Do not evaluate the integrals. You need not do any simplification.

**Solution:** Here is a picture of the region:

![Diagram of the region](image)

We will integrate with respect to \( x \), but we need two integrals. We solve for the \( x \)-coordinates of the intersections of the curves. First, set \( x^2 = (x - 2)^2 - 4 \), that is, \( x^2 = x^2 - 4x + 4 - 4 \). There is only one solution, namely \( x = 0 \). Second, set \( x^2 = (x - 4)^2 \), that is, \( x^2 = x^2 - 8x + 16 \). There is only one solution, namely \( x = 2 \). Finally, set \( (x - 2)^2 - 4 = (x - 4)^2 \), that is, \( (x^2 - 4x + 4) - 4 = x^2 - 8x + 16 \), or \( 4x = 16 \). There is only one solution, namely \( x = 4 \). Therefore the area is

\[
\int_0^2 (x^2 - [(x - 2)^2 - 4]) \, dx + \int_2^4 ((x - 4)^2 - [(x - 2)^2 - 4]) \, dx.
\]

3. (8 points.) Determine whether or not the improper integral \( \int_1^\infty (8 - 3 \cos(11y)) e^{-y} \, dy \) converges.

**Solution:** For all \( t \neq 0 \) we have \( -3 \leq -3 \cos(11y) \leq 3 \), so \( 5 \leq 8 - 3 \cos(11y) \leq 11 \). So

\[
0 \leq 5e^{-y} \leq (8 - 3 \cos(11y))e^{-y} \leq 11e^{-y}.
\]

In particular, \( 0 \leq (8 - 3 \cos(11y))e^{-y} \leq 11e^{-y} \). Using the substitution \( u = -y, \) so \( du = -dy \) and \( dy = -du \), we get

\[
\int e^{-y} \, dy = \int (-e^u) \, du = -e^u + C = -e^{-y} + C.
\]
So
\[ \int_1^\infty e^{-y} \, dy = \left[-e^{-y}\right]_1^\infty = \lim_{y\to\infty} (-e^{-y} + e^{-1}) = e^{-1} = \frac{1}{e}. \]

Thus \( \int_1^\infty e^{-y} \, dy \) converges. Therefore the Comparison Test shows that \( \int_1^\infty (8 - 3 \cos(11y)) e^{-y} \, dy \) converges.

Alternate solution (sketch): (This solution takes much more work, and is not recommended.) One can find the indefinite integral of \( \cos(11y) e^{-y} \) by integrating by parts twice, and then solving for \( \int \cos(11y) e^{-y} \, dy \).

Doing this gives
\[ \int (8 - 3 \cos(11y)) e^{-y} \, dy = -8e^{-y} + \frac{3}{122} e^{-y} \cos(11y) - \frac{33}{122} \frac{e^{-y}}{122^2} \sin(11y) + C. \]

Now \( \lim_{y\to\infty} e^{-y} = 0 \). Using the Squeeze Theorem, it follows that
\[ \lim_{y\to\infty} \cos(11y) e^{-y} = 0 \quad \text{and} \quad \lim_{y\to\infty} \sin(11y) e^{-y} = 0. \]

Therefore
\[ \int_1^\infty (8 - 3 \cos(11y)) e^{-y} \, dy = \left[-8e^{-y} + \frac{3}{122} e^{-y} \cos(11y) - \frac{33}{122} \frac{e^{-y}}{122^2} \sin(11y)\right]_1^\infty \]
\[ = \lim_{y\to\infty} \left[-8e^{-y} + \frac{3}{122} e^{-y} \cos(11y) - \frac{33}{122} \frac{e^{-y}}{122^2} \sin(11y)\right] \]
\[ = \left[-8e^{-1} + \frac{3}{122} \frac{e^{-1}}{122} \cos(11 \cdot 1) - \frac{33}{122} \frac{e^{-1}}{122} \sin(11 \cdot 1)\right] \]
\[ = \frac{1}{e} \left(8 - \frac{3 \cos(1)}{122} + \frac{33 \sin(1)}{122}\right). \]

In particular, the integral converges.

4. (12 points.) Find \( \int \frac{x}{(x+1)(2x+1)} \, dx \).

Solution: We do this integral using partial fractions. We need to find \( a \) and \( b \) such that
\[ \frac{x}{(x+1)(2x+1)} = \frac{a}{x+1} + \frac{b}{2x+1}. \]

Now
\[ \frac{a}{x+1} + \frac{b}{2x+1} = \frac{(2a+b)x + (a+b)}{(x+1)(2x+1)}. \]

If (1) is to be true for all real \( x \) for which the denominator is not zero, we must have
\[ 2a + b = 1 \quad \text{and} \quad a + b = 0. \]

The usual methods give the solution \( a = 1 \) and \( b = -1 \). So, using the substitutions
\( u_1 = x + 1 \quad du_1 = dx, \quad u_2 = 2x + 1, \quad \text{and} \quad du_2 = 2 \, dx, \)
we get
\[ \int \frac{x}{(x+1)(2x+1)} \, dx = \int \left( \frac{1}{x+1} - \frac{1}{2x+1} \right) \, dx \]
\[ = \int \frac{1}{x+1} \, dx - \int \frac{1}{2x+1} \, dx \]
\[ = \ln(|x+1|) - \frac{1}{2} \ln(|2x+1|) + C. \]
Alternate solution: This solution differs only in the method used to find $a$ and $b$ in the first solution. We have
\[
\frac{a}{x+1} + \frac{b}{2x+1} = \frac{a(2x+1) + b(x+1)}{(x+1)(2x+1)}.
\]
If (1) is to be true for all real $x$ for which the denominator is not zero, then
\[
(2) \quad x = a(2x+1) + b(x+1)
\]
for all real numbers $x$ except $x = -1$ and $x = -\frac{1}{2}$. By continuity, (2) must also hold when $x = -1$ and $x = -\frac{1}{2}$.
Putting $x = -1$ in (2) gives $-1 = a(2(-1) + 1)$, so $a = 1$. Putting $x = -\frac{1}{2}$ in (2) gives $-\frac{1}{2} = b(-\frac{1}{2} + 1)$, so $b = -1$.

5. (15 points.) Consider the region below the graph of $y = e^{-x^2}$ and in the first quadrant. It is rotated about the $y$-axis. Find the volume (possibly infinite) of the resulting solid.

Solution: Here are three pictures. On the right is a graph showing the lines and curve bounding the region to be rotated, going out only to $x = 3$; in the middle is a picture of part of the resulting solid of revolution, again going out only to $x = 3$; on the left is transparent version of this picture, with a cylindrical shell added (at $x = 1$).

We use cylindrical shells with vertical axis. For $x \geq 0$, the cylindrical shell at this value of $x$ and with thickness $\Delta x$ has radius $x$ and height $e^{-x^2}$, hence volume approximately $2\pi xe^{-x^2} \Delta x$. This leads to the integral
\[
\int_{0}^{\infty} e^{-x^2} \, dx.
\]
This is an improper integral. We first find the indefinite integral using the substitution $u = -x^2$, so $du = -2x \, dx$ and $2\pi xe^{-x^2} \, dx = -\pi \, du$:
\[
\int 2\pi xe^{-x^2} \, dx = \int (-\pi)e^{u} \, du = -\pi e^{u} + C = -\pi e^{-x^2} + C.
\]
Now
\[
\int_{0}^{\infty} 2\pi xe^{-x^2} \, dx = \left(-\pi e^{-x^2}\right)_{0}^{\infty} = \lim_{x \to \infty} (-\pi e^{-x^2} + \pi e^{-0}) = 0 + \pi = \pi.
\]

Alternate solution: (This solution takes somewhat longer.) We use horizontal disks. This can be done because the function $f(x) = e^{-x^2}$ has an inverse on $[0, \infty)$, namely the function $g(y) = \sqrt{-\ln(y)}$ for $y$ in $(0, 1]$. (Note that $-\ln(y) \geq 0$ since $0 < y \leq 1$.)

For $0 < y \leq 1$, a cross section at $y$ and perpendicular to the $y$-axis is a disk with radius $\sqrt{-\ln(y)}$. So its area is $\pi \left(\sqrt{-\ln(y)}\right)^2 = -\pi \ln(y)$. A disk at this position and with thickness $\Delta y$ (for small $\Delta y$) has volume $-\pi \ln(y) \Delta y$. This leads to the integral
\[
\int_{0}^{1} (-\pi \ln(y)) \, dy = -\pi \int_{0}^{1} \ln(y) \, dy.
\]
This is an improper integral. We first find the indefinite integral. Integrate by parts with \( u(y) = \ln(y) \), \( u'(y) = 1 \), \( v'(y) = y^{-1} \), and \( v(y) = y \). Then
\[
\int \ln(y) \, dy = \ln(y) \cdot y - \int y^{-1} \cdot y \, dy = y \ln(y) - \int 1 \, dy = y \ln(y) - y + C.
\]
Now
\[
\int_0^1 \ln(y) \, dy = (y \ln(y) - y) \bigg|_0^1 = 1 \cdot \ln(1) - 1 - \lim_{y \to 0^+} (y \ln(y) - y) = -1 - \lim_{y \to 0^+} (y \ln(y) - y).
\]
We have \( \lim_{y \to 0^+} y = 0 \). We rewrite
\[
\lim_{y \to 0^+} y \ln(y) = \lim_{y \to 0^+} \frac{\ln(y)}{\frac{1}{y}}
\]
In this form, it has the indeterminate form \( \frac{-\infty}{-\infty} \), so we can use L’Hopital’s Rule:
\[
\lim_{y \to 0^+} \frac{\ln(y)}{\frac{1}{y}} = \lim_{y \to 0^+} \left( -\frac{1}{y^2} \right) = \lim_{y \to 0^+} (-y) = 0.
\]
Putting everything together, we get
\[
\int_0^1 \ln(y) \, dy = -1 - 0 = -1.
\]
and the volume is
\[
-\pi \int_0^1 \ln(y) \, dy = \pi.
\]

6. (15 points.) A spiky pyramid-like solid has square horizontal cross sections. The total height is 2 meters, and for \( 0 \leq h \leq 2 \) the cross section at height \( h \) meters has side length \( (2 - h)^2 \) meters. Find the volume of this solid.

Solution: Here are two pictures of the object, to the left a solid version and to the right a semitransparent version.

![Diagram of spiky pyramid-like solid](image)

For \( 0 \leq h \leq 2 \), the horizontal cross section at height \( h \) has area \((2 - h)^2 = (2 - h)^4\). If a cross section at this position is fattened up to have thickness \( \Delta h \) (for small \( \Delta h \)), then it has volume \((2 - h)^4 \Delta h\).

Now consider a subdivision of \([0, 2]\) into \( n \) intervals, each of length \( \Delta h = 2/n \). (We don’t have to use equal length intervals; this is just for convenience.) For \( k = 1, 2, \ldots, n \) let \( h_k \) be a point in the \( k \)-th interval. Then the volume is approximately
\[
\sum_{k=1}^{n} (2 - h_k)^4 \Delta h.
\]
This looks like a Riemann sum for
\[ \int_0^2 (2 - h)^4 \, dh \]
So the volume is (using the substitution \( u = 2 - h \), so \( dh = -du \))
\[ \int_0^2 (2 - h)^4 \, dh = -\left( \frac{(2 - h)^5}{5} \right)_0^2 = \frac{32}{5} \]
Thus the volume is \( \frac{32}{5} \) cubic meters. (The units are required.)

7. (10 points.) A cable on the fourth moon of the planet Grogxth is 2 meters long and has linear density \( \frac{3}{1 + x^2} \) kilograms per meter at the point \( x \) meters from the heavy end. It is hanging from a hook in the basement of a building, with the heavy end at the top. Set up an integral which represents the amount of work needed to pull up the cable and lay it flat on the floor of the main level. The floor is one meter higher than the hook on which the cable hangs, and the acceleration due to gravity on the surface of the fourth moon of Grogxth is \( 4 \) m/sec\(^2\). Do not attempt to evaluate the integral.

**Solution:** Consider a short segment of the cable, with length \( \Delta x \) meters, and located \( x \) meters from the end with the hook. Its linear density is approximately \( \frac{3}{1 + x^2} \) kilograms per meter, so its mass is approximately \( \left[ \frac{3}{1 + x^2} \right] \Delta x \) kilograms. Its starting position is \( x \) meters below the ceiling, so it must be lifted \( 1 + x \) meters, against a downwards acceleration of \( g = 4 \) m/sec\(^2\). Therefore the work in Newton-meters is
\[ 4(1 + x) \frac{3}{1 + x^2} \Delta x. \]
By considering Riemann sums, we see that the total work is
\[ \int_0^2 4(1 + x) \frac{3}{1 + x^2} \, dx = \int_0^2 \frac{12(1 + x)}{1 + x^2} \, dx. \]

Other coordinate systems can also be used. If we let \( y \) be the distance from the floor of the main level, then the cable is between \( y = 1 \) (the heavy end) and \( y = 3 \). The work is
\[ \int_1^3 \frac{4y}{1 + (y - 1)^2} \, dy = \int_1^3 \frac{12y}{1 + (y - 1)^2} \, dy. \]
If we let \( h \) be the height above the floor of the main level, then the cable is between \( h = -1 \) (the heavy end) and \( h = -3 \). The work is
\[ \int_{-3}^{-1} 4(-h) \frac{3}{1 + (-1 + h)^2} \, dh = \int_{-3}^{-1} \left( -\frac{12h}{1 + (h - 1)^2} \right) \, dh. \]

8. (12 points.) Find \( \int \frac{1}{(4 + x^2)^{3/2}} \, dx \).

**Solution:** Use the substitution \( x = 2 \tan(\theta) \), with \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \). Thus, \( dx = 2 \sec^2(\theta) \, d\theta \), so
\[ \int \frac{1}{(4 + x^2)^{3/2}} \, dx = \int \frac{1}{(4 + 4 \tan^2(\theta))^{3/2}} \cdot 2 \sec^2(\theta) \, d\theta = \int \frac{2 \sec^2(\theta)}{(4 \sec^2(\theta))^{3/2}} \, d\theta. \]
We have \( \sec(\theta) > 0 \) since \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), so
\[
\int \frac{2 \sec^2(\theta)}{(4 \sec^2(\theta))^{3/2}} \, d\theta = \int \frac{2 \sec^2(\theta)}{8 \sec^3(\theta)} \, d\theta = \frac{1}{4} \int \cos(\theta) \, d\theta = \frac{1}{4} \sin(\theta) + C.
\]

Now
\[
\sin^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \frac{1}{\sec^2(\theta)} = \frac{\tan^2(\theta)}{1 + \tan^2(\theta)} = \frac{\frac{1}{4} x^2}{1 + \frac{1}{4} x^2} = \frac{x^2}{4 + x^2}.
\]

Since \( \sin(\theta) \) and \( x \) are either both positive or both negative, we get
\[
\sin(\theta) = \frac{x}{(x^2 + 4)^{1/2}}.
\]

Therefore
\[
\int \frac{1}{(4 + x^2)^{3/2}} \, dx = \frac{x}{4(x^2 + 4)^{1/2}} + C.
\]

(The last steps are necessary. However, a substantial amount of the credit will be given for getting to \( \frac{1}{4} \sin(\arctan(x/2)) + C \).)

9. (8 points.) Consider the region between the curve \( y = e^{-x} \), the lines \( x = 1 \) and \( x = 4 \), and the \( x \)-axis. It is rotated about the line \( y = -2 \). Write down a formula in terms of integrals for the volume of the resulting solid. Do not evaluate the integrals. You need not do any simplification.

**Solution:** Here are three pictures. On the right is a graph showing the lines and curve bounding the region to be rotated, as well as the axis of rotation; in the middle is a picture of the resulting solid of revolution; on the left is transparent version of this picture, with a washer shaped slice added (at \( x = 2 \)).

We use vertical washers. For \( 1 \leq x \leq 4 \), a cross section at \( x \) and perpendicular to the \( x \)-axis is an annulus with inner radius 2 and outer radius \( 2 + e^{-x} \). So its area is \( \pi (2 + e^{-x})^2 - \pi \cdot 2^2 \). An annulus at this position and with thickness \( \Delta x \) (for small \( \Delta x \)) has volume \( \pi(e^{-x})^2 - \pi(e^{-2x})^2 \Delta x \). This leads to the integral
\[
\int_1^4 \left( \pi(e^{-x})^2 - \pi(e^{-2x})^2 \right) \, dx.
\]

**Alternate solution:** Use cylindrical shells with axis parallel to the \( x \)-axis. Two integrals will be needed.

We use cylindrical shells with horizontal axis. We rewrite the equation of the curve as \( x = -\ln(y) \). The largest value of \( y \) on this curve is at \( x = 1 \), and is \( y = \frac{1}{e} \). We are thus interested in values of \( y \) between 0 and \( \frac{1}{e} \).

For \( 0 \leq y \leq \frac{1}{e} \), the cylindrical shell at this value of \( y \) and with thickness \( \Delta y \) has radius \( y \). It has radius \( y + 2 \). (Remember that we are rotating about the line \( y = -2 \).) If \( 0 \leq y \leq \frac{1}{e} \), it has length (height) \( 3 = 4 - 1 \), hence volume approximately \( 2\pi(y + 2) \cdot 2 \Delta y \). This leads to the integral \( \int_0^{1/e} 4\pi(y + 2) \, dy \). If \( \frac{1}{e} \leq y \leq \frac{1}{2} \), the shell has length (height) \( 4 - (-\ln(y)) = 4 + \ln(y) \). (Remember that \( \ln(y) < 0 \) since \( y \leq \frac{1}{e} < 1 \).) It therefore has
volume approximately $2\pi(y + 2)[4 + \ln(y)] \Delta y$. This leads to the integral $\int_{1/e^2}^{1/e^4} 2\pi(y + 2)[4 + \ln(y)] dy$. So the total volume is

$$\int_0^{1/e^4} 4\pi(y + 2) dy + \int_{1/e^4}^{1/e^2} 2\pi(y + 2)[4 + \ln(y)] dy.$$ 

EC. (20 extra credit points.) Find the volume of a four dimensional sphere of radius 2. Hint: Use a four dimensional version of one of the methods for finding volumes of three dimensional solids of revolution.

Significant credit will be given for getting the integral set up correctly.

Solution: There is no picture in this solution, since I don’t know how to draw a four dimensional sphere. You have to think of it as a souped up version of a three dimensional sphere.

We use the analog of the washer (or disk) method. Assume the sphere has center at the origin of a four dimensional coordinate system. We consider the three dimensional cross section in the direction perpendicular to the fourth axis, at distance $t$ units along the fourth axis. This cross section is a sphere of radius $\sqrt{2^2 - t^2} = \sqrt{4 - t^2}$. Therefore its three dimensional volume is $\frac{4}{3}\pi(4 - t^2)^{3/2}$. We imagine giving it a very small thickness $\Delta t$ in the direction of the fourth axis; the result has four dimensional volume $\frac{4}{3}\pi(4 - t^2)^{3/2} \Delta t$. Adding up the four dimensional volumes of such thin slices gives a Riemann sum for the integral

$$\int_{-2}^{-\frac{\pi}{2}} \pi(4 - t^2)^{3/2} dt.$$ 

This integral therefore represents the four dimensional volume of a four dimensional sphere of radius 2.

To compute this integral, we use the substitution $t = 2\sin(\theta)$, so $dt = 2\cos(\theta) d\theta$. The new limits of integration are $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. For these values of $\theta$, $\cos(\theta) \geq 0$, so

$$(4 - t^2)^{3/2} = (4 - 4\sin^2(\theta))^{3/2} = \left(\sqrt{4\cos^2(\theta)}\right)^3 = 8\cos^3(\theta).$$

(The point is that we never get $-8\cos^3(\theta)$.) Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{3}\pi(4 - t^2)^{3/2} dt = \frac{4}{3}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^4(\theta) \cdot 2 d\theta = \frac{64}{3}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta.$$ 

To evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta$, we use the formula $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ twice, once with $x = \theta$ and again with $x = 2\theta$. We get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}(1 + \cos(2\theta))\right)^2 d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos(2\theta) d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} \cos^2(2\theta) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos(2\theta) d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} (1 + \cos(4\theta)) d\theta.$$ 

Now

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} d\theta = \frac{\pi}{4},$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos(2\theta) d\theta = \frac{1}{4} \sin(2\theta) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0,$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} (1 + \cos(4\theta)) d\theta = \frac{\pi}{8} + \frac{1}{16} \sin(4\theta) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{8} + 0 = \frac{\pi}{8}.$$ 

Therefore

$$\int_{-2}^{-\frac{\pi}{2}} \frac{4}{3}\pi(4 - t^2)^{3/2} dt = \frac{64}{3}\pi \left(\frac{\pi}{4} + \frac{\pi}{8}\right) = 8\pi^2.$$
Comment: The volume of a four dimensional sphere of radius \( r \) is \( \frac{1}{2} \pi^2 r^4 \). For a five dimensional sphere, one gets something with \( \pi^2 \) as well, and for six and seven dimensional spheres one gets something with \( \pi^3 \).

Alternate solution: We use the analog of the cylindrical shell method. Assume the sphere has center at the origin of a four dimensional coordinate system. We take the axis of the shells to be the fourth axis. The shell at distance \( r \) from the center of the sphere has height \( 2\sqrt{2^2 - r^2} = 2\sqrt{4 - r^2} \). Its cross section is no longer a circle with radius \( r \), but rather the surface of a three dimensional sphere of radius \( r \), and has surface area \( 4\pi r^2 \).

We imagine giving it a very small thickness \( \Delta r \); the resulting four dimensional volume is then approximately

\[
4\pi r^2 \cdot 2\sqrt{4 - r^2} \Delta r = 8\pi r^2 \sqrt{4 - r^2} \Delta r.
\]

(The exact volume is

\[
\frac{4}{3} \pi (r + \Delta r)^3 \cdot 2\sqrt{4 - r^2} - \frac{4}{3} \pi r^3 \cdot 2\sqrt{4 - r^2} = \frac{8}{3} \pi \sqrt{4 - r^2} ((r + \Delta r)^3 - r^3)
\]

\[
= \frac{8}{3} \pi \sqrt{4 - r^2} (3r^2 \Delta r + 3r(\Delta r)^2 + (\Delta r)^3).
\]

The terms involving \((\Delta r)^2\) and \((\Delta r)^3\) are much smaller than the term involving \(\Delta r\), so may be ignored.) Adding up the four dimensional volumes of such thin shells gives a Riemann sum for the integral

\[
\int_0^2 8\pi r^2 \sqrt{4 - r^2} \, dr.
\]

This integral therefore represents the four dimensional volume of a four dimensional sphere of radius 2.

This integral can be done by the same trigonometric substitution as in the first solution. Also, it can be converted to the integral in the first solution by an integration by parts. Details are omitted.