Find the most general antiderivative of:

\( f(x) = 7 \) for all \( x \)  

\[ \text{answer: } F(x) = 7x + C \]

Comments: MI sample HW: "all or nothing" part graded, with not scattered (incomplete) comments.

Solutions are posted.

Also: \( \sqrt{t} = \sin(\sqrt{t}) \) does have an elementary antiderivative, [not easy to find]

\( k(x) = x^{12} \) for all \( x \)  

Answer was: \( \frac{1}{13} x^{13} + C \)

We got this: expect answer to involve \( \int x^{12} \)

\[ \int x^{12} = 13 \cdot x^{12} \]

Fix: this: take \( K(x) = \frac{1}{13} x^{13} + C \)

This idea works only because 13 is a constant.

\( l(x) = \cos(x) \)  

\[ \text{Answer: } L(x) = \sin(x) \]

\( m(x) = 2x \cos(x^2) \)  

\[ \text{Answer is } \left( \frac{d}{dx} (-\sin(x)) = -\sin(x) \right) \]

\[ M(x) = \sin(x) \]

\[ + \left( \frac{d}{dx} (x^2 \sin(x^2)) = 2x \sin(x^2) + x^2 \cos(x^2) \cdot 2x \right) \]

Check: By the chain rule, \( M'(x) = \sin(x) \cdot \frac{d}{dx} (x^2) = \cos(x^2) \cdot 2x \). Yes

Try \( n(x) = \cos(x^2) \).

What happened if we try\[ \frac{1}{2x} \cdot \sin(x^2)? \]

\[ \frac{d}{dx} \left( \frac{1}{2x} \cdot \sin(x^2) \right) = \frac{d}{dx} \left( \frac{1}{2} x^{-1} \sin(x^2) \right) \]

\[ = -\frac{1}{2} x^{-2} \sin(x^2) + \frac{1}{2} x^{-1} \cos(x^2) \cdot 2x \]

\[ = -\frac{\sin(x^2)}{2x^2} + \cos(x^2) \cdot 2x \]

This is not wanted.

Unwanted extra term: we got it because 2x is not a constant.

The trick for \( x^{12} \) works only if you are off by a constant multiple!

Thus, no antiderivative, but no "elementary" formula for it.
Suppose you are given a function $Q$ such that $Q'(t) = e^{t^2}$ for all $t$.

Find an antiderivative of $p(x) = 2 \cos(x) e^{\sin^2(x)} = 2 \cos(x) \exp\left(\sin^2(x)\right)$.

Take $P(x) = 2 Q(\sin(x))$. Check: $P'(x) = 2 Q'(\sin(x)) \cdot \sin'(x)$

$= 2 e^{\sin^2(x)} \cos(x)$.

$Q'(t) = e^{t^2}$ for any $t$.

So $Q'(i) = e^{i^2} = e^{-1}$.

$Q'(b) = e^{b^2}$.

$Q'(w+i) = e^{(w+i)^2}$.

$Q'(e^x) = e^{(e^x)^2} = e^{e^{2x}}$.

$Q'(\sin(x)) = e^{[\sin(x)]^2} = e^\sin^2(x)$.

Caution: do not confuse $Q'(\sin(x))$ with $\frac{d}{dx} Q(\sin(x)) = Q'(\sin(x)) \cdot \sin'(x)$.

Principle here: recognizing the output of the chain rule.

Approximate area under graph between $x = 0$ and $x = 4$.

$\int_0^4 e^{\sin^2(x)} \cos(x) \, dx$

Better approximation: sum of areas of red rectangles, which is

$\sqrt{0.1} + \sqrt{1.1} + \sqrt{2.1} + \sqrt{3.1} = 0.31 + 1.21 + 2.21 + 3.21$.

Can we do better? Sum of areas of green rectangles:

$\sqrt{0.2} + \sqrt{1.2} + \sqrt{1.2} + \sqrt{2.2} + \sqrt{3.2} + \sqrt{3.2} + \sqrt{4.2}$.

Better approximation: sum of areas of red rectangles, which is

$\sqrt{0.1} + \sqrt{1.1} + \sqrt{2.1} + \sqrt{3.1} = 0.31 + 1.21 + 2.21 + 3.21$.
For $f(x)$ in place of $\sqrt{x}$, use with 4 sub-intervals:

$$f(0) \cdot \frac{1}{1} + f(\frac{1}{2}) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2}.$$ 

With 8 sub-intervals:

$$f(0) \cdot \frac{1}{2} + f(\frac{1}{2}) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} + f(\frac{3}{2}) \cdot \frac{1}{2}.$$ 

To do better: Could use 16 sub-intervals.

Could use 1027 sub-intervals or 229 sub-intervals, etc.

Under good conditions, the limit as $n \to \infty$ of the sum of areas one gets using $n$ sub-intervals will exist, and be the exact area under the curve.