Ex. Find an antiderivative \( F \) of the function \( f(x) = x^2 + 2x + 7 \) such that \( F(3) = 6 \).

Start with general antiderivative: \( F(x) = \frac{1}{3}x^3 + x^2 + 7x + C \). Using:

\[
6 = F(3) = \frac{1}{3}(3^3) + 3^2 + 7(3) + C = 39 + C \Rightarrow C = -33 \text{ and \thus:\} } \\
F(x) = \frac{1}{3}x^3 + x^2 + 7x - 33.

We approximate the area with the

\[
f(0)(\frac{1}{2}) + f(\frac{1}{2})(\frac{1}{2}) + f(1)(\frac{1}{2}) + f(\frac{3}{2})(\frac{1}{2}) + \\
+ f(\frac{5}{2})(\frac{1}{2}) + \sqrt{\frac{1}{2} (\frac{1}{2})} + \sqrt{\frac{3}{2} (\frac{1}{2})} + \sqrt{4 (\frac{1}{2})}.
\]

Values of the function. Used here values at left endpoints.

Call this a left hand Riemann sum.

Right hand Riemann sum. Using right endpoints of the subintervals

\[
f(\frac{1}{2})(\frac{1}{2}) + f(1)(\frac{1}{2}) + f(\frac{3}{2})(\frac{1}{2}) + f(\frac{5}{2})(\frac{1}{2}) + \\
+ \sqrt{\frac{3}{2} (\frac{1}{2})} - \sqrt{\frac{1}{2} (\frac{1}{2})} - \sqrt{4 (\frac{1}{2})}.
\]

Since the particular function is increasing, the left sum is an underestimate
and the right sum is an overestimate.

Could use midpoints. With 8 equal length subintervals, would get:

\[
f(\frac{1}{4})(\frac{1}{2}) + f(\frac{3}{4})(\frac{1}{2}) + f(\frac{5}{4})(\frac{1}{2}) + f(\frac{7}{4})(\frac{1}{2}) + \\
+ \sqrt{\frac{3}{4} (\frac{1}{2})} + \sqrt{\frac{5}{4} (\frac{1}{2})} + \sqrt{\frac{7}{4} (\frac{1}{2})}.
\]

Randomly chosen points in the corresponding subintervals

\[
f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right).
\]

\langle length of subinterval \rangle.
General Riemann sum:

\[ f(b_1)(x_1-x_0) + f(b_2)(x_2-x_1) + \ldots + f(b_n)(x_n-x_{n-1}). \]

\( \Delta x_k \) is the length of the k-th subinterval, \( t_k \) is in the k-th subinterval \( [x_{k-1}, x_k] \).

Rectangle height \( f(t_k) \Delta x_k \). Add up these areas.

If the limit as length of the longest subinterval goes to zero of these Riemann sums (for arbitrary \( t_k \)), exists, then \( f \) is \underline{integrable} on \([a, b]\), and call the limit \( \int_a^b f(x) \, dx \) or \( \int_a^b f \).