MATH 252 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4.

This homework is due on Canvas on **Wednesday** 27 January 2021 (week 4), at 10:00 pm.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and use correct notation. (See the web page on notation.) Be sure your Canvas submission is a single file, and that the file name contains no spaces, parentheses, or other disallowed characters.

1. (10 points.) Let $f$ be a continuously differentiable function defined for all real numbers. Suppose $f(0) = 1$, $f(1) = -3$, $f(2) = -5$, $f(3) = 6$, $f(4) = -11$, $f(5) = -3$, $f(6) = 5$, $f(7) = -2$, $f(10) = 18$, and $f(11) = 100$.

What is $\int_3^5 f'(2x + 1) \, dx$? (You will not need all the information provided.)

**Solution:** We use the substitution $u = 2x + 1$. Thus, $du = 2 \, dx$, so $dx = \frac{1}{2} \, du$. When $x = 3$ we have $u = 7$, and when $x = 5$ we have $u = 11$. So

$$\int_3^5 f'(2x + 1) \, dx = \int_7^{11} \frac{1}{2} f'(u) \, du = \frac{1}{2} f(11) - \frac{1}{2} f(7) = \frac{100}{2} + \frac{2}{2} = 51.$$  

**Alternate solution:** We do the indefinite integral using the substitution $u = 2x + 1$. Thus, $du = 2 \, dx$, so $dx = \frac{1}{2} \, du$, and

$$\int f'(2x + 1) \, dx = \int \frac{1}{2} f'(u) \, du = \frac{1}{2} f(u) + C = \frac{1}{2} f(2x + 1) + C.$$  

(Or else just observe that the function $g(x) = \frac{1}{2} f(2x + 1)$ satisfies $g'(x) = f'(2x + 1)$.) Therefore

$$\int_3^5 f'(2x) \, dx = \frac{1}{2} f(2 \cdot 5 + 1) - \frac{1}{2} f(2 \cdot 3 + 1) = \frac{1}{2} f(11) - \frac{1}{2} f(7) = \frac{100}{2} + \frac{2}{2} = 51.$$  

2. (10 points.) Find

$$\frac{d}{dx} \left( \int_1^{2x} \cos(t^2) \, dt \right).$$

There should be no integral signs in your answer.

**Solution:** The function to be differentiated is $F(g(x))$, where

$$F(x) = \int_0^x \cos(t^2) \, dt \quad \text{and} \quad g(x) = 2x.$$  

Now $F'(x) = \cos(x^2)$ by the Fundamental Theorem of Calculus, so the chain rule gives

$$(F \circ g)'(x) = F'(g(x))g'(x) = \cos((2x)^2) \cdot 2 = 2 \cos(4x^2).$$

Since the derivative is supposed to be a function of $x$, any answer containing the variable $t$ is automatically wrong.

3. (10 points.) Let $H$ be a function such that $H'(t) = 2 \sin(5t^2)$ for all real $t$. Find

$$\int \sin(t) \sin \left( 5 \cos^2(t) \right) \, dt$$

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in terms of elementary functions and \( H \).

**Solution:** We use the substitution \( u = \cos(t) \). Thus, \( du = -\sin(t) \, dt \), so \( \sin(t) \, dt = -du \), and

\[
\int \sin(t) \sin \left( 5 \cos^2(t) \right) \, dt = - \int \sin(5u^2) \, du = - \frac{1}{2} \int 2 \sin(5u^2) \, du = - \frac{1}{2} H(u) + C = - \frac{1}{2} H(\cos(t)) + C.
\]

4. (10 points.) Let \( f \) be a continuously differentiable function defined for all real numbers. Suppose

\[
\int_0^2 f(x) \, dx = 10, \quad \int_0^5 f(x) \, dx = -31, \quad \int_2^5 f(x) \, dx = -21, \quad f(0) = -3, \quad f(2) = -1, \quad f(5) = 20, \quad f'(0) = -81, \quad f'(2) = -200, \quad f'(5) = 1001.
\]

What is \( \int_2^5 xf'(x) \, dx \)? (You will not need all the information provided.)

**Solution:** Integrate by parts, taking

\[
u(x) = x, \quad v'(x) = f'(x), \quad u'(x) = 1, \quad \text{and} \quad v(x) = f(x).
\]

Thus,

\[
\int_2^5 xf'(x) \, dx = xf(x) \bigg|_2^5 - \int_2^5 f(x) \, dx = 5f(5) - 2f(2) - (-21) = (5)(20) - 2(-1) + 21 = 123.
\]

5. (10 points.) Let \( F \) be a function such that \( F'(t) = e^{-2t^3} \) for all real \( t \). Find

\[
\int tF(t) \, dt
\]

in terms of elementary functions and \( F \).

**Solution:** Integrate by parts, taking

\[
u(t) = F(t), \quad v'(t) = t, \quad u'(t) = e^{-2t^3}, \quad \text{and} \quad v(t) = \frac{1}{2} t^2.
\]

Thus,

\[
\int tF(t) \, dt = \frac{1}{2} t^2 F(t) - \int \frac{1}{2} t^2 e^{-2t^3} \, dt.
\]

We do the remaining integral using the substitution \( w = -2t^3 \). Thus, \( dw = -6t^2 \, dt \), so \( t^2 \, dt = \frac{1}{6} \, dw \), and

\[
\int \frac{1}{2} t^2 e^{-2t^3} \, dt = \int \left( \frac{1}{2} \right) \left( -\frac{1}{6} \right) e^w \, dw = -\frac{e^w}{12} + C_0 = -\frac{e^{-2t^3}}{12} + C_0.
\]

So (with \( C = -C_0 \))

\[
\int tF(t) \, dt = \frac{t^2 F(t)}{2} - \int \frac{1}{2} t^2 e^{-2t^3} \, dt = \frac{t^2 F(t)}{2} + \frac{e^{-2t^3}}{12} - C_0 = \frac{t^2 F(t)}{2} + \frac{e^{-2t^3}}{12} + C.
\]

Just writing the last line shows nearly enough work.